How to generate “good mental habits” in resolution of Calculus problems through experiments with technology?

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Abstract

In this paper, some examples are given of the interrelation between intuitive and analytic thought in teaching and learning of mathematics with technology. It is illustrated how the theory of dual processes in cognitive psychology can be used as theoretical framework for the study and analysis of several didactic situations. The influence of experimentation, validation of conjectures and use of technology are studied in the context of intuitive versus analytic thought. We analyze how to use technology to lead students to arrive to correct conclusions by analyzing their answers from different points of view. Methods are suggested for generation of “good mental habits” in the resolution of mathematics problems with technology. The examples teach the student the necessity of a formal proof based in processes related to *S2* system. We show how the new technologies can contribute to the education of intuition related to *S1* processes as well as the construction of “bridges” between both types of processes.

1. Introduction

“*Learning begins with actions and perception, proceeds hence to words and concepts and should end in good mental habits. This is the general aim of mathematics teaching, to develop in each student as much as possible the good mental habits of tackling any kind of problem*”. G. Polya [1]

Students often arrive at Calculus course without a background in abstract reasoning and with a limited experience in application of mathematical techniques. In order to acquire the necessary knowledge and the appropriate development of abstract reasoning; the students need first to gather evidence of mathematical phenomena.

Following experimental trend in Calculus we encourage students to formulate and to verify conjectures, to discover patterns, to explore, to make a synthesis of computational results and to appreciate proof and experiments as functional tools.

Implementation of experimental methods change the way we teach calculus, imposing new demands to methodology, evaluation and curricula that should finally develop new abilities in the student which in the past were not considered relevant, such as - inductive thought, pattern’s discovery, mathematical intuition, critical analysis of experimental results, construction and validation of conjectures and experimental search for formal proof.

The transposition [2] into mathematical education of this new approach is not exempt of obstacles as J. B. Lagrange outlines in [3]. In several activities of our didactic experiences in Calculus’s teaching we have built laboratories, exercises and examinations using technology. However, the answers we hoped to obtain from the students, frequently become in something totally unexpected. These obtained discrepancies often could be explained through important disarrangement between students' intuition and the requirements of formal system of mathematics.

“*Intuition comes to us much earlier and with much less outside influence than formal arguments which cannot really understand unless we have reached a relatively high level of logical experience and sophistication…*”, [4].

The resolution of mathematical problems is a complex and demanding mental activity and often involves several associate tasks. People who are occupied in such activities are much more likely to respond to some of these tasks by blurting out whatever comes to mind. According to Kahneman and Tversky [5] the rationality of thought is bounded by certain heuristic shortcuts which are applied in resolution of complex tasks and that in certain cases can lead to systematic errors.

We argue that cognitive psychology establishes a useful theoretical framework for the study and analysis of several didactic situations that appear during the implementation of some mathematical tasks. We use in this paper the so termed *Dual Processes Theory* [5, 6] to analyze, on the base of examples, the interrelation between intuitive and analytic thought in mathematical problem – solving situations in Calculus class and how this often conflicting relationship can be affected by the introduction of new computational tools like symbolic calculators.

According to Dual Process Theory, our cognition and behaviour operate in parallel in two quite different modes, called System 1 (*S1*) and System 2 (*S2*), roughly corresponding to our common sense notions of intuitive and analytical thinking.

Besides the typical errors by carelessness (mistakes in algebraic manipulations, sign's change, etc.) or by simple ignorance of the subject matter, there is other kind of errors, that students do, which could be related to the way in which the *S1* and *S2* systems work.

Due to the lack of experience and knowledge the students give solutions to a mathematical problem with a fast and intuitive answer typical of the *S1* system, without the controls and regulations that are characteristic of the *S2* system.

Many authors are pointed out the persistence of student's misconceptions with respect to specific topics and task [7]. Tirosh and Stavy in [8] state they have observed that students react in a similar way to a wide variety of conceptually non related problems which share some external common features. This fact allowed them to suggest that many response that literatures describe as alternative conceptions (misconceptions) could be explained as evolving from common intuitive rules as “More of A – More of B”, “Same A – Same B”, “Everything can be divided”, “Overgeneralised linearity”, etc.

Often student’s intuition about certain concepts and ideas are not in line with accepted scientific frameworks in which are based the operation principles of computational tools used in mathematical experiments. In order to assure an intelligent dialogue with the computational tool is necessary to impose a correct definition of objects and concepts used in mathematics. For example, student should clearly understand concepts such as list, vector, matrix, expression, function, equation, etc to appropriately work with a sophisticate calculator, like Classpad300.

To consider an effective transposition of experimental methods to teaching and learning Calculus we need on one hand a better understanding of the impact that new technologies produce and on the other hand a careful study of mental process involved during the resolution of mathematical problems. We should worry not only about mathematical knowledge that student should possess, but also on meta-cognition tools he needs to manage. It is fundamental to know and to understand in a better way the natural heuristic strategies of thought when it faces resolution of complex tasks, as it is a mathematical problem.

The use of computational tools requires that cognitive functions should be distributed in an optimal way between student and tool. In the old paradigm, teaching mathematics was based on a repetitive practice until perfecting certain knowledge. With the use of technology this practice should be change in an intelligent way.

2. Some previous experiences.

In [2] J. B. Lagrange analyzes one example of erroneous answer given by students in an experimental mathematics task, which was reported by S. Pozzi in [6]:

"*... Pozzi reports on an observation of two students who were trying to find a general rule for differentiating a product of a polynomial with trigonometric function. Asking DERIVE to differentiate* $\cos(x)\cdot(7x^3+2x)$ *they got* $(21x^2+2)\cos(x)-x(7x^2+2)\sin(x)$. *They concentrated on the central part* $\cos(x)-x(7x^2+2)$ *which they found very similar to the initial expression. They tried to induce a general rule involving the transformation of a product into a difference. Of course the central part has no meaning because it is not a sub-expression of the derivative. But, to the students, it appeared to be the key to finding a rule because it is perceptually close to the initial expressions. Student's algebraic knowledge about the structure of the expressions was not strong enough to counterbalance this perceptual evidence and they could not make good use of the Derive´s help…."*

As conclusion he states

"*... It is clear that we have to reflect on the prior algebraic knowledge required. Students do not necessarily need strong procedural abilities but obviously should not be lacking some key knowledge of algebraic structure*".

In the experience analyzed by J. B Lagrange, the students not only ignored the basic principles of algebraic structures, but rather they didn't also adopt a critical position before giving their answer. They threw the first thing that came to mind without thinking for an instant that their answer could be mistaken. Clearly this error was not due to the use of technology or implementation of experimental methods; rather with their help its existence was detected. If the student usually do not practice critical thinking or if he lack the mechanism for examining his associations and determining whether they make sense in a given situations then frequently we will hear from him meaningless answers.

In the former situation the obtained DERIVE' expression evoked in the student's mind certain associations. Since they lack understanding of the topic, they cannot examine these associations and know whether they constitute a correct answer or not.

All of us have associations when we are faced certain situations. In general, we cannot control our associations because they are internal reactions to external stimulus, but we can control our behaviour, the external reaction to the stimulus. S. Vinner in [9] refers to thought processes related with this kind of situations as a pseudo-conceptual behaviour and states that:

"*A particular instance of pseudo-conceptual behaviour is not an indication of somebody's intellectual ability. It is only a characterization of a particular thought process that occurred in somebody's mind. It is a mode of thinking. Of course, if the pseudo-conceptual behaviour is quite frequent, it does indicate the quality of thinking of the person involved*"

In the next section we will explain on the base of *Dual Process Theory* that frequently, student's lack of experience and knowledge produces this kind of situations.

3. The theory of dual processes and its application to teaching and learning mathematics.

The distinction among intuitive and analytic way of thought is clearly established in cognitive psychology in the so called "*Dual-Process Theory*". In [12] U. Leron and O. Hazzan, writes:

"According to this theory, our cognition and behaviour operate in parallel in two quite different modes, called System 1 ***S1*** *and System 2* ***S2****, roughly corresponding to our common sense notions of intuitive and analytical thinking. These modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins (****S2*** *being evolutionarily more recent and, in fact, largely reflecting cultural evolution) … Like perception,* ***S1*** *processes are characterized as being fast, automatic, effortless, unconscious and inflexible (hard to change or overcome); unlike perception,* ***S1*** *processes can be language-mediated and relate to events not in the here and-now (i.e., events in far-away locations and in the past or future). In contrast,* ***S2*** *processes are slow, conscious, effortful and relatively flexible. The two systems differ mainly on the dimension of accessibility: how fast and how easily things come to mind."*

An important feature of this classification in cognitive psychology is that in the processes linked to *S2* system are included monitoring functions. These functions belong to the effortful operations of *S2* system and monitor the activities of *S1* system. In the context of mathematical problem - solving situations we will associate these monitoring functions with check-up of the way in which problems are solved.

When a student erroneously answers to a question, or at least not in the way that it is expected, in general we blame the *S2* processes of his failure. However, it is probable that the self - regulation mechanisms of *S2* system has not had time of playing any role at all. The students often give erroneous answers in spite of having the enough knowledge and abilities (*S2* related) to give a correct solution. The *S1* processes are so fast that are immediately manifested. The problem is sometimes greater, when the student ignores important data from the exercise and assumes that his answer is correct without any intent to check-up not only the answer itself but the way in which the problem was solved.

"*People are not accustomed to thinking hard, and are often content to trust a plausible judgment that quickly come to mind*", D. Kahneman [5].

Strengthen the processes linked to *S2* system is what the mathematical education and science in general have done, nevertheless seems to be required a better understanding of the interrelation between both systems in teaching and learning mathematics. A better understanding of the interrelation of the processes *S1* and *S2* will allow us, without doubts, to communicate to our students suitable heuristic strategies in the resolution of mathematical problems or what G. Polya called “good mental habits”. The mathematics is, first of all, to know how to do, is a science where the method usually is more important that the content.

A suggestion given by the authors U. Leron and O. Hazzan in [12] is the following:

“It is need to train people to be aware of the way S1 and S2 operate, and to include this awareness in their problem-solving toolbox”

3.1 The Polya's proposal from the point of view of the Kahneman - Tversky theory.

The Polya’s work constituted an important motivation for one of current approaches in mathematical research, the so called *experimental mathematics* [13]. Following the experimental line in mathematics education we would like to highlight in this subsection that Polya's proposition for mathematical problem – solving strategies encourages an effective use of natural heuristics of thought.

Indeed, Polya's model [14, 18] proposes a set of four phases with some questions and suggestions that constitute a guide for the search and exploration of answers alternatives. In summary, the phases are the following ones: 1) *Understand the problem*; 2) *Find the connections between the data and the unknown quantities*; 3) *Make and execute a plan* and 4) *Examine the obtained solution.*

In the step 1 Polya stand out the important role of analytic thought related to *S2* system to appropriately understand the problem before proceeding to solve it. In the step 2 he outlines the use of auxiliary, similar, or simpler problems related with the main problem, but more accessible. According to Kahneman [15] often when we try to solve a complex problem happens what he denominates “*attribute substitution*”, which consists in that judgements are mediated by heuristic, when an individual values a specific real attribute of the object of judgement by means of another heuristic attribute which comes easier to mind. The real attribute is less accessible and another related attribute which is more available replaces the first one. This associative relation with real attribute is so close and fast that monitoring functions which are characteristic of the *S2* system cannot be activated and who is solving the problem doesn't notice that he is really responding to another question. Some of the former mentioned intuitive rules can be explained with this simple concept. In fact, to be aware of this fact and avoid possible mistakes on this stage, Polya includes in the step 1 the understanding of the problem, (using *S2* related process). Only with this condition it is possible to use in an appropriate way the substitution of attributes which is characteristic of the *S1* system so that start to work in searching for auxiliary problems that are more accessible and whose solutions in principle are known by the one who is trying to solve the problem. Also in the step 2, Polya proposes to enunciate the problem in another form, to create in this way the effect that Kahneman denominates “*anchoring effect*” in which judgement is influenced by a temporal increase of accessibility of some particular value of real

attribute, i.e. Polya proposes to change formulation of the problem to highlight other aspects which had possessed a low accessibility in the previous formulation.
Finally in the steps 3 and 4 Polya encourage critical reasoning regarding the resolution of the problem, highlighting this way the role of monitoring mechanisms of the system *S2*.

4. Some examples.

In words of Isaac Newton, "*Examples are better than precepts for learning the arts*". Therefore, to better understand how *S1* and *S2* processes work in the resolution of mathematical problems and how those processes can be affected by the introduction of new technologies, we include in this section some examples of didactic situations experienced at Universidad Diego Portales and Pontificia Universidad Católica de Chile [20,21,24,25], in which the use of graphic calculators as Casio CFX-9850, Algebra FX-2.0 and Classpad300 were implemented in the cycle of calculus courses for engineering. Those calculators provided us with a set of useful applications and with a basic programming language [16]. This implementation has generated an interesting debate related to the influence of technology in different aspects in the teaching and learning of calculus.

4.1 Example 1.

Several contributions related to the teaching and learning of the concept of limit have shown difficulties in understanding this concept no only due to its richness and complexity but also because the cognitive aspects cannot be generated purely from its mathematical definition (see for example [17]). The following didactic experience happened during the task resolution about the existence of limits of several variables functions in a multivariable Calculus laboratory [21]. Some students gave an incorrect conclusion about tendency of limits, in particular when they analyzed sequences that led to results seemingly dissimilar. The students used their intuition and responded in a fast and wrong way, without giving to analytical thought an opportunity to manifest it. The task was:

Activity 4.1.1: *Use a table of numerical values for the function* $f(x,y)$ *near the origin and make a conjecture regarding its limit, when* (x,y) *tends to* $(0,0)$*:*

1. $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$
2. $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$

   a. *Prove mathematically your conjectures.*

Solution

For the first limit it was considered the following sequences:

$$\left(x_n, y_n\right) = \left(\frac{1}{2^n}, \frac{k}{2^n}\right); \text{ with } k = 1, 3, 5.$$

These sequences converge to $(0,0)$ when $n \to \infty$. Convergence of the following sequence was studied:

$$f\left(x_n, y_n\right) = \frac{x_n{}^2 - y_n{}^2}{x_n{}^2 + y_n{}^2}$$

when n tend to infinity.

Using the calculator in the "Main" menu we define the function $f\left(x, y\right) = \dfrac{x^2 - y^2}{x^2 + y^2}$ and in the "Sequence" menu we define a sequence for each k value, as it is shown in figure 1

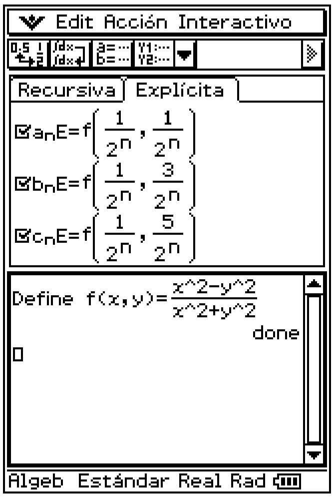

Figure 1

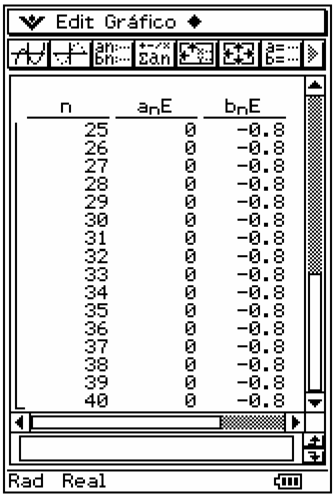

Figure 2

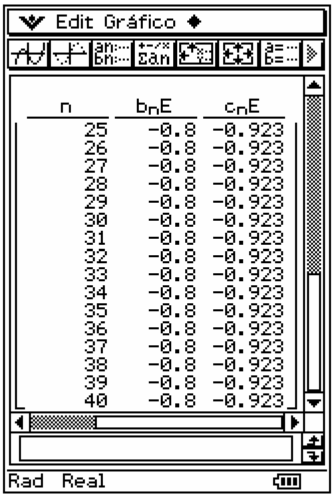

Figure 3

Results with at least three decimals digits of accuracy are shown in figures 2 and 3.

After taking $n = 1 \ldots 40$ in the sequence, students concluded clearly that sequences converge to different limits depending on the k values and this only fact it's evidence indicating that limit (1) does not exist.

For the second limit the same sequences were used. We only redefined the function in the "Main" menu. Results between $n = 85$ and $n = 100$ are shown in figures 4 and 5.

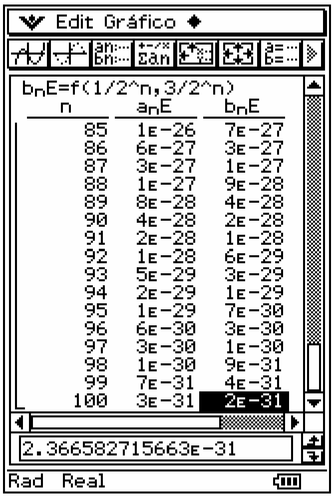

Figure 4

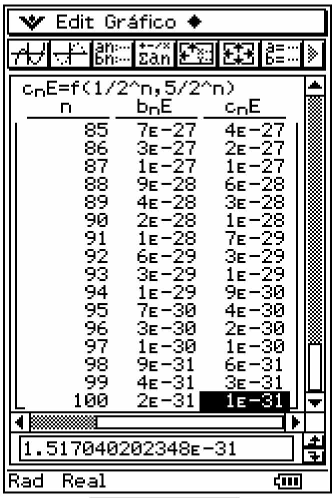

Figure 5

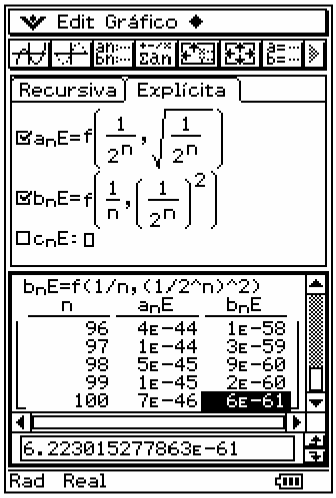

Figure 6

Some students concluded that this limit doesn't exist either, seeing that sequences after taking $n = 1\ldots100$ apparently converged to different values.

In the answer given by students for the limit (2) probably existed, among other phenomena, *effects of context on accessibility* [5, 15], since in the former case (1) the limit indeed doesn't exist.

The students' intuition, based on *S1* processes, assured to them that for the second limit just happened the same thing as in the previous case and they assumed their answer as the correct one, without a critical analysis that would necessarily involve *S2* processes, arriving in this way to a wrong conclusion. Students never doubted about their answer. In intuitive reasoning, uncertainty is poorly considered. The given intuitive answer rarely establishes several options because in most cases only a single option comes to mind. The options that were rejected are not represented. Doubt is a phenomenon of *S2* system and it is related to the ability to think incompatible thoughts about the same thing.

The use of calculator in this case was of great utility. Several actions were carried out with its help. The value of n was increased and the numbers in the table were compared again. The numbers were written in several formats which allowed us to resolve doubts raised from the scientific notation used in the tables of figures 4 and 5, and were compared involved order of magnitude, relating it to infinite concept. In this way a suitable interpretation of numbers in this numerical experiment allowed us to arrive to a correct answer.

After this error was corrected, we emphasized the important fact that even though that the used sequences converge to zero, independently of the k values, this is still not a conclusive evidence, because it is not known if for other sequences of type $(x_n, y_n) \underset{n\to\infty}{\to} 0$ convergence to zero will be conserved. The figure 6 shows calculations for the following sequences: $(x_n, y_n) = \left(\frac{1}{2^n}, \sqrt{\frac{1}{2^n}}\right)$ and $(x_n, y_n) = \left(\frac{1}{n}, \left(\frac{1}{2^n}\right)^2\right)$.

These results suggest that limit (ii) exists and it is zero, but this is only a conjecture but not a proof of this fact. Through this numeric experiment students noticed the necessity of a formal proof. There was not any inconvenience in continuing devising new sequences to check that the limit always tend to zero with greater or smaller speed in its convergence, but continuing in this way would not have any sense at all. In Polya's words:

"In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting".

This type of didactic situations obtained by using technology is an illustration of what we meant by intelligent dialogue with computational tools. For this case was built the following formal proof:

To prove that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ doesn't exist, we consider the substitution $y = k \cdot x$ that gives $\lim\limits_{x\to 0} \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$

The limit depends on the k values and therefore doesn't exist. The k value dependence geometrically means that the value of the limit depends on the directions of convergence to $(0,0)$.

Let us proof the existence of limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$ using the ε - δ language. We know that:

- $\sqrt{x^2+y^2} < \delta \Rightarrow \sqrt{y^2} < \delta \Rightarrow |y| < \delta$. On the other hand $\left|\frac{x^2 y}{x^2+y^2}\right| < |y|$, since $\frac{x^2}{x^2+y^2} < 1$. Therefore $\left|\frac{x^2 y}{x^2+y^2} - 0\right| < |y| < \delta$, thus taking $\delta = \varepsilon$, we have $\left|\frac{x^2 y}{x^2+y^2} - 0\right| < \varepsilon$ and this proves that limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$ exists and it is zero.

4.2 Example 2.

A quite common error in mathematics is to associate points of graphic intersection of curves with points obtained by equating equations of these curves. Although this is true in Cartesian coordinates it is not always true for polar coordinates, where it should be distinguished among the idea of two curves that "cut" on a point with the idea of two curves that just "cross". That is, curves which pass through the same point but in different moments.

It is a common teaching practice to prepare examples of problems where student's prediction based on *S1* process will probably be wrong (see for example [10, 11]). The former misconception can be related with the intuitive rule "same A - same B" studied in [8] and we can anticipate the students' erroneous answer to plan the instruction for its remediation. With the use of technology we recommend to students compare their predictions with the calculator or computer answer before a final conclusion should be accepted. In this way we could be able to stimulate monitoring role of *S2* process. Knowledge of this kind of learning situations enables researchers and teachers to plan appropriate sequences of instruction.

The following example illustrates this situation:

Activity 4.2.1: *Consider the following curves in polar coordinates:*

$$r = \cos(\theta), \quad r = \sin(\theta)$$

These equations represent two circles (see figure 7). Determine the intersection points.

Solution

The figure 7 shows that both circumferences intersect on two points, where one of them is the pole (coordinate origin). The question is: How to find all intersection points? or otherwise, for which values of θ do both circumferences intersect?. The answer offered by students is that these two points can be obtained if the equations of both curves are equated. Solving the equation $r = \cos(\theta) = \sin(\theta)$ we find that $\theta = \frac{\pi}{4} + k\pi$, with k integer. For $k = 0$ we obtain one of the intersection points, given by $(r,\theta) = \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

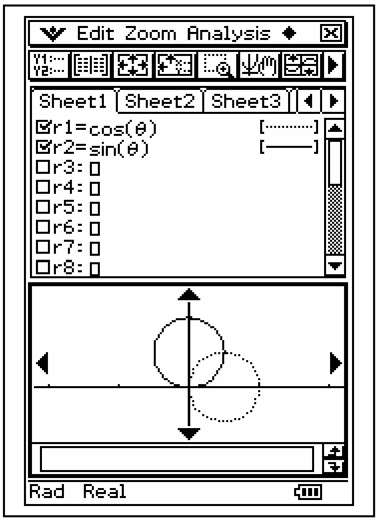

Figure 7

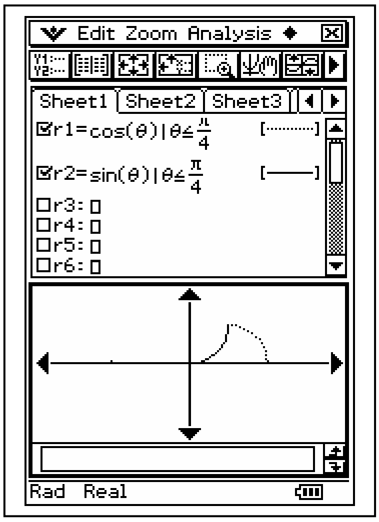

Figure 8

Students and even some teachers expressed, using their intuition (*S1* related processes) that for some other value of k the intersection point situated at the pole should be obtained. A bigger effort was required to show that none of possible solutions obtained by equating curve's equations contain the pole, for any value of k. This fact allowed motivating students to understand the dynamics of curve construction in polar coordinates. The technology in this case constituted a great help. The curves graphs were built again, but this time was shown not only the final result, but also the drawing process for the curves. Indeed, it was noticed that the graph of the curve $r = \cos(\theta)$ has the starting point one unit to the right of the pole and moves counterclockwise from the polar axis until arriving to the intersection point of the two curves for $\theta = \frac{\pi}{4}$. On the other hand, the curve $r = \sin(\theta)$ has its starting point in the pole and also moves counterclockwise from polar axis until arriving to the intersection point of the two curves. In fact, the graph of curve $r = \cos(\theta)$ just arrives to the pole when $\theta = \frac{\pi}{2}$ Figure 8 shows what happen for the graphics of both curves when θ takes values from 0 to $\frac{\pi}{4}$. That is why intersection point of both curves corresponding to the pole doesn't appear in the solution of the equation obtained equating the two curves, simply because these curves pass through the same point but in different instants. We should notice that technology "helps" to detect all intersection points ("cut" and "cross") of polar curves that would be in many cases difficult to find without use of technology. Let us think, for example in the possibility of more complicated polar equations.

As an additional comment we should also notice that calculus teachers often come from the pure mathematics area where the idea of speaking with terms as "movement" or "time" was usually rejected. In fact, definitions like those of limit of a function by means of epsilon-delta language were created to avoid conceptions associated to other area's terminology different from pure mathematics. It is very possible that this tendency toward the purification in the educators formation has impeded too many of them (or it has taken them away from the idea) to think of curves in polar coordinates as curves coming from the "movement" of points depending of "instants".

4.3 Example 3

The following activities [24] were applied to students of Calculus with knowledge on analytic geometry, algebra and trigonometry; also to a group of university mathematics teachers with no prior experience using technology for educational purposes. The requested activities were:

Activity 4.3.1 Obtain the graphic of the function, given in polar coordinates $r(\theta) = 3\cos(\theta) + 2\sin(\theta)$*, using classpad300 with a standard visualization window and determine to what known curve corresponds.*

Solution The obtained graph is given in figure 9

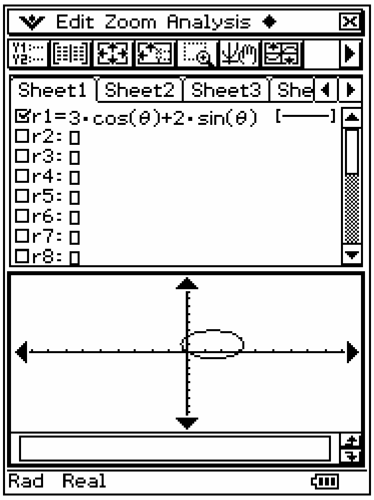

Figure 9

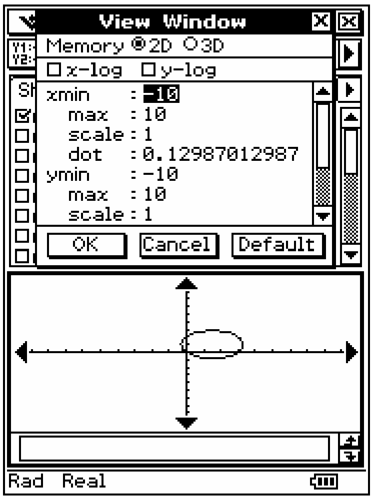

Figure 10

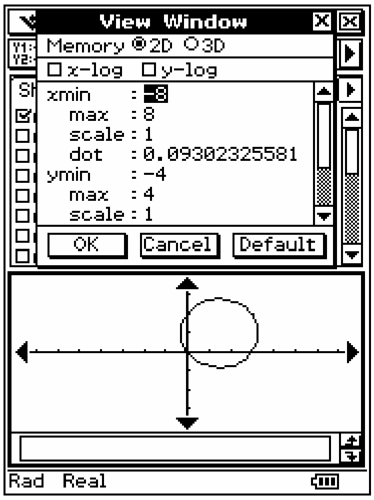

Figure 11

The almost unanimous opinion of all, teachers and students, was that the given polar function represents an ellipse. As a next step was suggested to revise the visualization window used to obtain this graphic (see figure 10).

It was communicated by the expert that the scale of visualization window not corresponds to a 1:1 scale in relation to the calculator screen. It was informed that relationship between screens width and high of graphic calculators is usually 2:1, that is, screen width double its high. Therefore, if we want to see a 1:1 scale graphic in calculator screen the units assigned to X - axis should be twice of the units assigned to Y - axis. A new graph is requested, but now with a visualization window given in figure 11.

The participants, students and teachers, doubt this time of their previous conclusion and conjecture that the drawn graphic is really a circumference.

Also, the expert made notice a strange fact that although the graphic of the curve is complete, the calculator continued calculating and it delayed a little more time in finishing. This fact worried them because they didn't know how to determine the cause of this additional time seemingly unnecessary used in the construction of the graphic.

It is requested to participants to mathematically prove this new conjecture:

Activity 4.3.2: *Prove that the curve is indeed a circumference.*

Few participants were able to suggest how to prove this conjecture. The expert suggested transforms polar equation $r = 3\cos(\theta) + 2\sin(\theta)$ to a cartesian one. Students were faced to the following challenges:

a. "Note" that it's convenient to multiply both side of the equation by r (most suggested to square the equation without noting the possible later algebraic difficulties).
b. Use transformation equations relating polar and cartesian coordinates.
c. Apply successively to each variable x and y the operation of completing the square.

Only some of participants were able to carry out the following routine stating from the equation $r = 3\sin(\theta) + 2\cos(\theta)$:

- $3r\cos(\theta) + 2r\sin(\theta) = r^2$
- $3x + y = x^2 + y^2$
- $x^2 - 3x + y^2 - 2y = 0$
- $\left(x - \frac{3}{2}\right)^2 + (y-1)^2 = \frac{13}{4}$

Finally participants recognized that the resultant equation is indeed a circumference with centre in $\left(\frac{3}{2}, 1\right)$ and radius $\frac{\sqrt{13}}{2}$.

One more task was required to graphically corroborate the participants' conclusion:

Activity 4.3.3: *Use calculator in* "Conics" *menu to build a graphic of a circumference with before deduced characteristics and compare the resulting graphic with the one obtained previously for polar function.*

Participants obtained the graphic given in figure 12

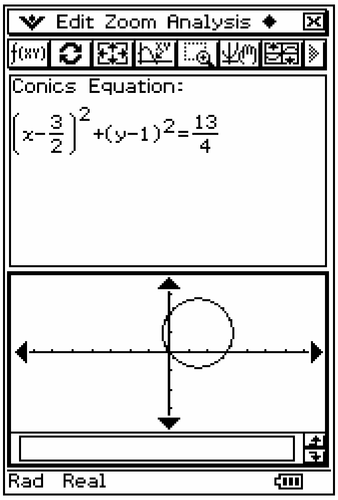

Figure 12

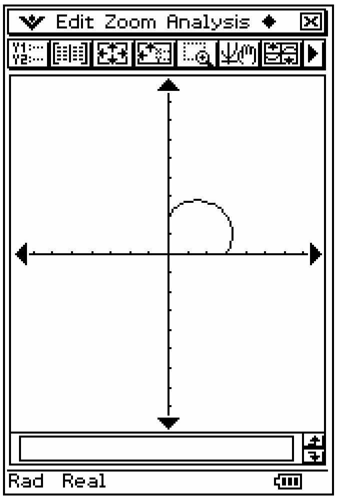

Figure 13

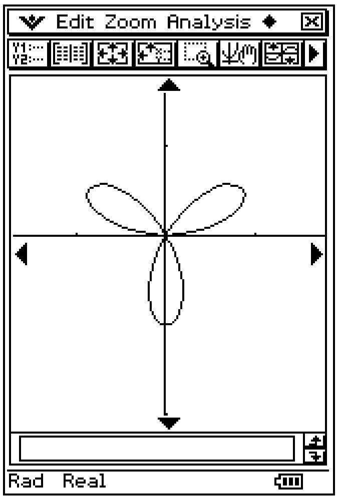

Figure 14

Finally it is requested students to explain the reason for the apparent calculator's delay to build graphic of the curve in polar coordinates. Participants were not able to determine that the problem is due to an excessive assignment given to the variable θ. To explain this fact it is suggested to revise the visualization window once again, in particular assignment of values for the variable θ. The participants noted that this variable takes values in the interval $[0, 2\pi]$. However, they didn't note that because circumference is not centred in the origin, its graphic could be complete, assigning to θ values in the interval $[0, \pi]$.

Previous experience highlighted that most of the students didn't understand how to work with polar coordinates. They didn't recognize the fundamental differences between the concepts of orderly couple (r, θ) in polar coordinates and orderly couple (x, y) in cartesian coordinates.

To improve understanding of how functions are represented in polar coordinates, the following task was given.

Activity 4.3.4: *Which should be the optimal interval for θ values, if we want to obtain the graph represented in the figure 13?*

This challenge shows a better performance of participants, most of them achieved to establish appropriate interval for θ, that is $\theta \in \left[0, \frac{\pi}{2}\right]$.

The use of graphic calculator, in these examples allowed us to detect the necessity to develop the following good habits:

a. Strengthen the work's habit of constantly relate what is graphically observed with its critical intellectual interpretation (in connection with the ellipse and circumference confusion).
b. Develop interest in determining causes of observed real or supposed calculator's errors.
c. Stimulate the ability of search for a scientific answer to the challenges, correlating graphics and algebraic meanings. In particular develop the ability to correlate cartesian graphic conceptions with polar perspectives.

This last point has shown to be a very big challenge for the student. A typical example of this kind of problem consists on recognizing the meaning of a "negative" value for r in a polar graphic. For example, Why we only need values between 0 and π to have a complete graphic for $r = \sin(3\theta)$? (See figure14). Student doubts about the presence of a petal in a "negative" area of the cartesian plane, in circumstance that θ has taken values in the first and second quadrant, that is, arguments that give "positive" values for the sinus function.

4.4 Example 4.

We will show an example [24, 25] in which technology demands special new mental habits if we want to improve the understanding of concepts as *parameter and variable*, and at the same time we want to involve new epistemic challenges [22]. To prepare animation with calculators demands a comprehensive understanding of mathematical concepts and powers scientific spirit of research.

4.4.1 Classical definition of "limit of a function"

We remember the definition of a "limit of a function":

$$\lim_{x \to a} f(x) = L \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\,[0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

Let's suppose we want to make an animation of that definition, for instance, with the function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

When x approaches to 1, we want to see that $f\left(x\right)$ approaches to 2. In this case δ is equal to the given ε because:

$$\left|\frac{x^2 - 1}{x - 2} - 2\right| = \left|\frac{(x+1)(x-1)}{x-1} - 2\right| = \left|x - 1\right| < \delta = \varepsilon$$

Note that to get a model to animate the definition with a graphic calculator as Casio CFX 9850 GB Plus, or CFX 9860 or Algebra FX-2.0 we will have to use rectangular equations as well as parametric equations. In fact we will need:

a. A function **Y1** representing the function $f\left(x\right)$

b. Functions **Y2** and **Y3** to produce $0 < \left|f(x) - 2\right| < \varepsilon$ to mean $2 - \varepsilon < f(x) < 2 + \varepsilon$

c. Vertical lines for which we will have to use parametric equations for $1 - d < x < 1 + d$

The following figures show how the equation has to been introduced in the Graph Window of calculator Casio Algebra FX-2.0 Plus.

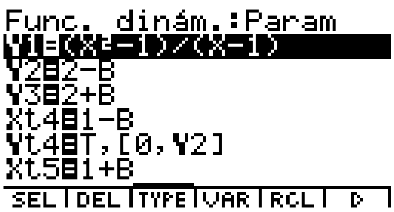

Figure 15

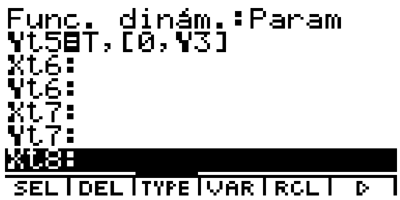

Figure 16

It is important to note that **Y2** and **Y3** work as dependent variables as well as constants. That is the idea of a ***parameter.*** Let's see how the animation works on this calculator:

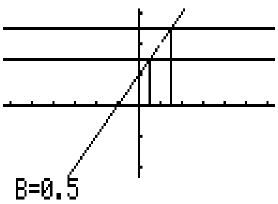

Figure 17

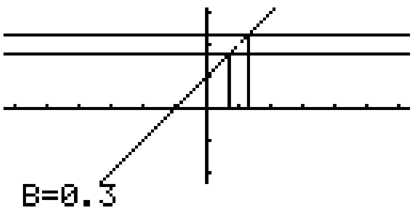

Figure 18

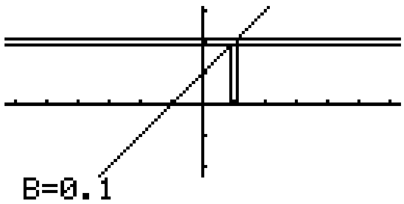

Figure 19

4.4.2 An interesting case: what happens when $f(x)$ is not linear?

For instance, $\lim_{x\to 2} x^2 = 4$. In this case δ is not equal to ε. The usual classical techniques to obtain a δ is the following: $|x^2-4| = |(x-2)(x+2)| < \delta|x+2|$. Let $\delta \le 1$. As $|x-2| < \delta$ we have that $|x-2| < 1$.Therefore $-1 < x-2 < 1$ and so $3 < x+2 < 5$, that is $-5 < 3 < x+2 < 5$ and therefore, $|x+2| < 5$. So, $|x^2-4| < 5\cdot\delta$. So, we have $\delta = \min\left(1, \frac{\varepsilon}{5}.\right)$

Students usually do questions related to this kind of proof:

(1) Why did we choose $\delta \le 1$?
(2) Is it actually necessary to introduce that kind of restriction over δ in order to get a proof?
(3) Could have been chosen another value to limit the value of δ? In that case, which would have been the main changes in the proof?

Teachers find difficult to explain and give answers as the following ones:

(1) Clearly for a given ε, the existence of δ is not unique. In fact there exist infinitely δ. And once one δ is found then each δ less that δ we have found will be useful.
(2) The *biggest possible* δ can be obtained but it is difficult to get it.
(3) The election of $\delta \le 1$ is due to algebraic manipulating reasons. However, in many cases other bounded values for δ can be and it is sometimes convenient to obtain.

So, the question is: In which way can technology help teachers to explain all those points?

Our plan included the following objectives:

a) How to find always the *biggest possible* δ
b) Trying to find the *biggest possible* δ involves new challenges as the necessity of solving different kind of difficult equations.
c) In resolving different kind of equations involves the necessity of teaching some numerical methods and creating special programs for the calculator.
d) With the biggest δ it is possible to create animations for calculators in a easier way.

In our example of $\lim_{x\to 2} x^2 = 4$ we can actually find the biggest δ because we will get just a second degree equation in which δ is easily obtained by hand.

In fact, if we do not bound δ then we obtain: $|x^2-4| = |(x-2)(x+2)| < \delta|x+2|$. As $|x-2| < \delta$ we have that $-\delta < x-2 < \delta$.

Therefore $-\delta+4<x+2<\delta+4$ and $-\delta-4<-\delta+4<x+2<\delta+4$, that is, $-(\delta+4)=-\delta-4<-\delta+4<x+2<\delta+4$ that is equivalent to $|x-2|<\delta+4$.Therefore we have:

$$|x^2-4|<\delta|x+2|<\delta(\delta+4)=\varepsilon \Rightarrow \delta^2+4\delta-\varepsilon=0 \Rightarrow \delta=\frac{-4\pm\sqrt{16+4\varepsilon}}{2} \Rightarrow \delta=-2\pm\sqrt{4+\varepsilon}$$

In this way we found that the biggest possible δ will be $\delta=-2+\sqrt{4+\varepsilon}$

The animation using the variables and parameters adequately involves the following syntax in a calculator as Casio Algebra FX-2.0:

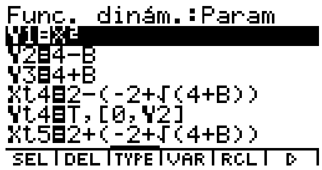

Figure 20

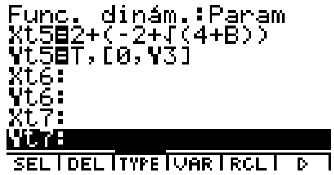

Figure 21

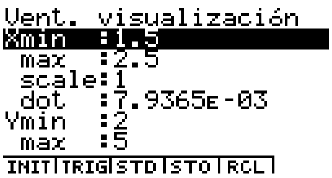

Figure 22

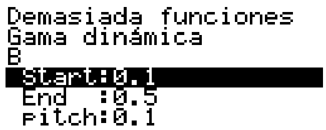

Figure 23

The animation uses the biggest δ and shows the following:

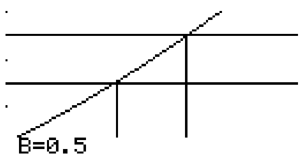

Figure 24

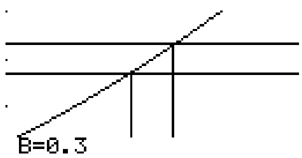

Figure 25

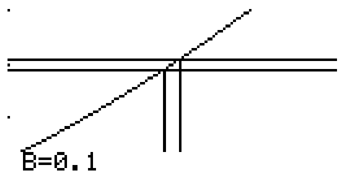

Figure 26

If it is not possible to get by hand the solution of an $\delta-\varepsilon$ equation then we will need numerical methods to solve the equation, as Bisection Method, Newton's Method or Secant Method. For the Bisection Method we have the following Program:

```
======BISECT R======
Y=Type↵
"FUNCION"?→Y1↵
Lbl 0↵
"A"?→A↵
A→X↵
Y1→G↵
```

```
======BISECT R======
"B"?→B↵
B→X↵
Y1→H↵
If (G×H)≤0↵
Then Goto 1↵
Else Goto 0↵
```

```
======BISECT R======
Lbl 1↵
0.5×(A+B)→C↵
A→X↵
Y1→D↵
C→X↵
If (D×Y1)≤0↵
```

```
======BISECT R======
Then Goto 2↵
Else C→A↵
Goto 3↵
IfEnd↵
A↵
Lbl 2↵
```

```
======BISECT R======
Lbl 2↵
C→B↵
B↵
Lbl 3↵
C◢
Goto 1
```

Figure 27

For the Newton's Method we have the following Program:

```
======NEWTON 2======
"FUNCION"?→Y1↵
"X"?→X↵
For 1→K To 10↵
X-Y1/d/dx(Y1,X,0.01)→
X↵
Next↵
```

```
======NEWTON 2======
"X"?→X↵
For 1→K To 10↵
X-Y1/d/dx(Y1,X,0.01)→
X↵
Next↵
X
```

Figure 28

For the Secant Method we have the following Program:

```
======SECANTE ======
"FUNCION"?→Y1↵
"A"?→A↵
"B"?→B↵
Lbl 1↵
A→X↵
Y1→Y↵
```

```
======SECANTE ======
B→X↵
X-Y1×((B-A)/(Y1-Y))→T
↵
B→A↵
T→B↵
B◢
```

```
======SECANTE ======
↵
B→A↵
T→B↵
B◢
Goto 1↵
```

Figure 29

4.5 Example 5.

An important mental habit in the teaching-learning process when implementing computer algebraic systems are related with the understanding of the interrelation existent between results obtained by hand and those obtained from technology.

The following situation appeared when trying to determine $\int \sec(x)\,dx$. It is known that the result of this indefinite integral is $\ln\left|\sec(x)+\tan(x)\right|$. In fact, this can be proved by amplifying $\sec(x)$ by $\sec(x)+\tan(x)$. Consider $u=\sec(x)+\tan(x)$, then we have $du=\sec(x)\tan(x)+\sec^2(x)$ and therefore:

$$\int \sec(x)\,dx = \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}dx =$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln\left|\sec(x)+\tan(x)\right| + C$$

However, when students try to obtain $\int \sec(x)\,dx$ as a result of a symbolic calculator as Classpad300, they have to introduce the expression $\frac{1}{\cos(x)}$ instead of $\sec(x)$, because calculators do not work with trigonometric functions as $\sec(x)$, $\operatorname{cosec}(x)$ and $\operatorname{cotan}(x)$. The calculator gives the following result:

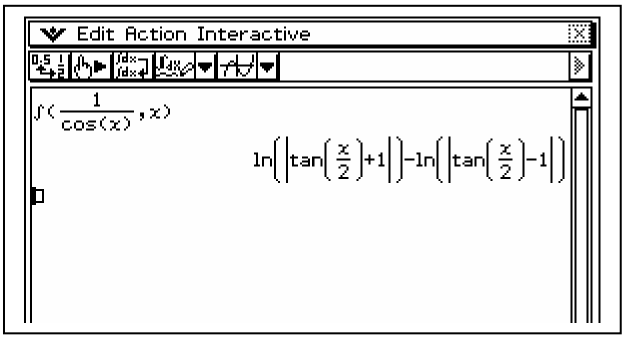

Figure 30

The first attitude of some students is trying to prove that both results are equivalent, that is:

$$\ln\left|\sec(x)+\tan(x)\right| \;=\; \ln(\tan\left(\frac{x}{2}\right)+1) - \ln(\tan\left(\frac{x}{2}\right)-1)$$

But in a second attitude they use to ask for the reason why the calculator gave such a different kind of result including half angles, instead of giving an answer more similar to the one obtained by hand. For instance, they expected the calculator to give $\ln\left|\frac{1}{\cos(x)}+\tan(x)\right|$.

According to our experience, never happened that a student (and in many circumstances even experienced teachers), realized that the calculator was programmed with the half angle substitution that are used when trying to calculate rational expressions containing sinus and cosines functions. In fact, if we consider $\frac{1}{\cos(x)}$ as one of that kind of expressions we can see that using half angle substitution $\tan\left(\frac{x}{2}\right)$ or $x = 2\arctan(u)$ we get $dx = \frac{2}{1+u^2}du$ and considering $\cos(x) = \cos(2\frac{x}{2}) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1$ together with the relation obtained through an auxiliary right angle triangle for $\tan\left(\frac{x}{2}\right)$(considering opposite side to $\frac{x}{2}$ as ***u*** and adjacent side to $\frac{x}{2}$ as **1)**, we can get $\cos\left(\frac{x}{2}\right)$ in terms of *u*, that is: $\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+u^2}}$. So we have: $\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1 = 2\frac{1}{1+u^2} - 1 = \frac{1-u^2}{1+u^2}$. Therefore:

$$\int \frac{1}{\cos(x)}dx = \int \frac{1+u^2}{1-u^2}\frac{2}{1+u^2}du = \int \frac{2}{1-u^2}du = 2\int\left(\frac{1}{2(1+u)} + \frac{1}{2(1-u)}\right)du =$$

$$= \ln|u+1| - \ln|u-1| + C = \ln\left|\tan\left(\frac{x}{2}\right)+1\right| - \ln\left|\tan\left(\frac{x}{2}\right)-1\right| + C$$

getting the same result as the calculator.

It is interesting to state that after this experience some students were motivated and decided to expand the idea and tried to get a general recursive formula for $\int \sec^n(x)\,dx$ in terms of $\int \sec^{n-1} dx$ by obtaining results in terms of half angles, instead of formulas that usually appear in tables in terms of complete angles. In that way a good use of technology in the process of teaching-learning calculus motivated students to have a new behaviour and a new special mental habit in studying integral calculus and in particular the methods of finding primitives.

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