IS A RECTANGLE A PARALLELOGRAM? – TOWARDS A BYPASS OF VAN HIELE LEVEL 3 DECISION MAKING

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In this paper we wish to share our thoughts and results of a pilot study about the possibility of overcoming the gap between the geometrical level of thinking (van Hiele) that is expected in given assignments and students' level of thinking. Our pilot study formulated a method that aims to enable students who did not reach van Hiele's level 3 (Order) to classify special parallelograms via conceptual behavior. Korean ninth-grade low achievers were introduced to parallelograms in a process involving dynamic and continuous shape changes of multi-named-objects in order to enlarge their concept image. The results of the study seemed promising.

BACKGROUND

Acquiring geometry concepts and coping with geometry assignments are a serious pitfall for average and low achievers worldwide. This research focused on the difficulties in classifying parallelograms and endeavoured to enable students to correctly classify special parallelograms through conceptual understanding.

A pilot research held in Korea provided data on average to slow Korean 9th grade students' understanding of quadrilaterals, a topic studied in the 8th grade. Findings from the pilot suggested that students had difficulty classifying "special" parallelograms (rectangle, square and rhombus) where inclusion relations are needed. However, most students in the pilot were able to correctly define a parallelogram.

Researchers aspired to help students overcome difficulties in inclusion tasks, where there is potential gap between students' geometrical thinking level (van Hiele, 1986) and task level. For example, the classification of a rectangle as a parallelogram requires van Hiele's level 3 geometrical thinking (Order), hence students who fail to classify it correctly do not seem to have reached this level.

Our study suggests an instructional approach to overcome this difficulty, which was pilot tested. We wish to get a better interdisciplinary, theoretical understanding of the process of acquiring these visual geometrical concepts (Fischbein & Nachlieli, 1998) with their inter-relations, and to study the cognitive and perceptive perspectives. Such an understanding might shed light on both practical and theoretical perspectives.

METHOD AND RESULTS

Stage I: Detecting and analyzing the difficulties

As part of this research 137 average-slow Korean 9th graders were asked to classify basic geometry shapes on a research-designed questionnaire and provide argumentations. Results were analyzed and the lowest achievers were identified. For Example, one of the students centered on in this paper, Nua, completed the required task by mistakenly classifying a concave quadrilateral and a trapezoid as non-quadrilaterals. Her explanation for the concave shape was that *"the number of vertices is three, so it is not a quadrilateral"* (van Hiele level 2 argumentation) and her explanation for the trapezoid was that *"because it has a name (a trapezoid), it is not a quadrilateral"* (level 1 argumentation).

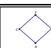
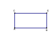
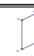
												
Number of mistakes (N=137)	3	20	28	22	22	6	7	16	23	29	10	14
% of mistakes (integers)	2	14	20	16	16	4	5	11	16	21	7	1

Table 1: Average-slow Korean 9th graders classifying shapes as parallelograms/non-parallelograms (Mistakes over 15% are highlighted.)

Students were also asked to decide whether 12 provided drawn shapes were parallelograms or not. Table 1 introduces the shapes and the main difficulties of the answerers.

As aforementioned, after analysis of answers to problems on the questionnaires researchers identified students who failed to classify shapes correctly. A number of them, including Nua, were asked to take part in the intervention. Hereinafter this paper will focus exclusively on the case of Nua.

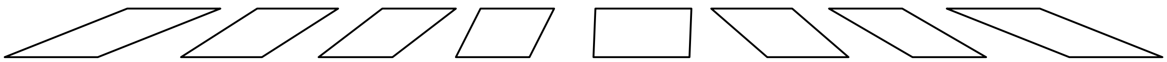
Stage II: Intervention

Procedure. The intervention took place at school, and it was videotaped and transcribed. One of the researchers, a non-speaker of Korean, guided the intervention (in English) and the other translated English to Korean and vice versa.

The intervention was targeted at suggesting to the students a new experience in which they might broaden their current concept image (Tall & Vinner, 1981) of special parallelograms using various senses. It was carried out through a *directed orientation* (Hoffer, 1983) process in which the student was active and reflective about shapes in their dynamic presentation. The teacher in this case was also one of the researchers.

The intervention was based on three stages (see the intervention plan in the Appendix):

1. Formation of visually-continuous members of a family of shapes, using demonstrations and gestures**.** Twisting continuously dynamic parallelograms (e.g. changing the angles of shapes made of stripes) provides a family of shapes where special cases (e.g. rectangle and square) become "natural" members in this category. This process of slow, smooth, and continuous changes, verbalizing explicitly the property of continuity seen in the set of shapes, may contribute to the natural acceptance of "special shapes" (e.g. rectangle among parallelograms) as members of the same family.



2. Posing explicitly the possibility of naming a shape with two names (e.g. rectangle as well as parallelogram) which might cause a cognitive conflict, and providing examples of everyday life cases in which Multi-names are used for the same object (e.g. Someone can be called by his first name, by his nationality etc.)**.** Through this we try to convince the students to accept the legitimacy of having two (or more) names for the same geometrical object as well. For instance, it is possible to name a shape a rectangle in addition to parallelogram. The former name is an outcome of the *static prototypical global perception* of the figure and the other name is a result of the shape's *being a member of a visually-continuous family*.

Note that the suggested intervention does not intend to cause a "jump" into level 3 geometrical thinking, using definitions and logical argumentation. The transfer from level 2 to level 3 is neither trivial, nor a must. Even with extensive instruction one may not be able to proceed to a higher level of geometrical thinking. Van-Hiele himself suggested a five-phase instructional process for this "upgrade" to occur. But even if it is adopted by the teacher, one cannot be sure of this kind of progress. That is where our suggestion comes in. Instead of relying on moving into a higher level, this approach suggests a "bypass" in which level 1-2 geometrical thinking may provide a conceptual, informal argumentation followed by correct answers. That is, the classification of shapes is based on *understanding* and not on *memorization*.

3. Using the same procedure of building continuous sets of shapes for various cases:

- Using a set of parallelograms built from the same sides, continuously changing their angles from one parallelogram to the other, including the rectangle.
- Using a set of parallelograms with one pair of sides remaining the same and the other pair continuously changing from one parallelogram to the other (keep angles equal respectively in all shapes, including the rhombus.)
- Using a set of rhombuses, with continuous change in their angles, including the square.
- Using set of equilateral triangles, all with the same sides, increasing its "base" continuously, including the isosceles triangle.

The case of Nua

The above intervention was applied to several Korean students; one of them was Nua, 9th grade student. Before the intervention, on stage 1, Nua considered shapes to be parallelograms whenever their 4 sides were equal (she might have meant two pairs of opposite sides) and angles in the shapes were not 90 degree (see 1st questionnaire on Table 2). She testified that shapes were considered to be parallelograms when (a) *"4 sides were parallel"* (probably meaning two pairs of opposite sides) or when *"the lengths of all four sides are same but it has no right angle"* (Researchers suspected she suffered from some confusion about the properties of parallel and equal - "a package of properties" after Gal & Linchevski, 2000 - which was found to be true in a later interview with the student.) (b) the shape had no other known name (i.e. special parallelogram). Her reasoning in this case was to quote its name, claim that even though the sides are parallel it is a… [name], or just say it is not a parallelogram. These kinds of argumentations suggest level 1-2 reasoning.

Then, when she joined the intervention (stage 2), she was first asked to classify *paper-cut* shapes into parallelograms/non parallelograms. Nua immediately divided the shapes into the two groups, 5 out of 11 answers were incorrect. Observing the resulting groups, Nua looked doubtful about her classification and changed the place of two shapes (see Table 2, where multiple answers appear for the same shape). She hesitated classifying the "almost square" and a "turned" parallelogram (physically its two longer sides vertically) as parallelograms, so she relocated them. As earlier explained, her classification of a shape as a parallelogram was based on shapes having two pairs of equal and parallel sides (level 2 argumentation) while non-parallelograms were shapes with other names (level 1 argumentation).

When asked to point at figures which have the properties of parallelograms but also have another name she pointed to the square. Her response to the question, *"Is that a parallelogram?"* was *"No!"* This reasoning and decision-making is a typical level-2 answer in which the student is able to analyze the properties of shapes but judgment is based on the properties of prototype.

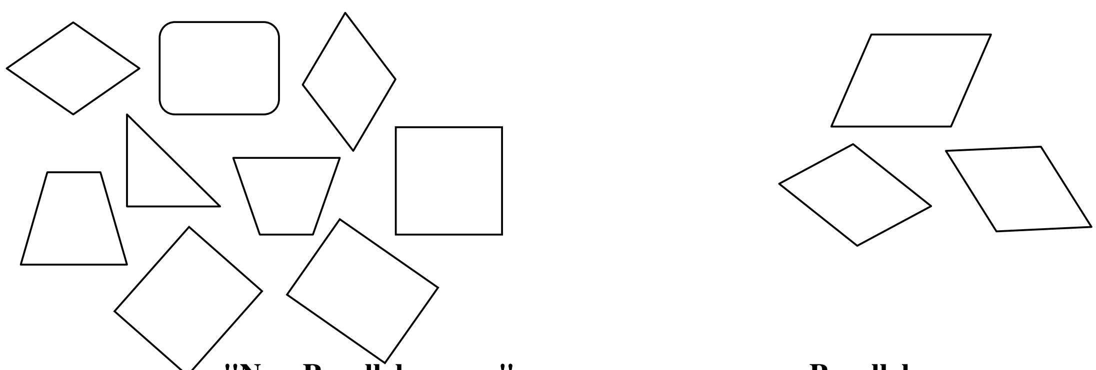
At this point it may be concluded that the student had incorrect ideas about what should be included in the groups - this requires level 3 thinking - while her reasoning can be characterized as level 1-2. This situation might be considered a "dead end" unless the student acquires level 3 thinking or goes through rote learning.

The following dialogue can give more evidence to her way of thinking:

Teacher: Looking at a shape, generally speaking (the teacher kept all shapes on the table hidden) how do you decide if the shape is a parallelogram or not? What are your criteria? If you encountered a shape in which its opposite sides were equal but it had more than one name, how would you decide?

Student: I would decide after looking at the shape.

To summarize, at this stage Nua was able to correctly judge all *paper-cut* **prototypes**. She could not yet judge correctly any of the **non-prototype** shapes, i.e. those which could not "become" parallelograms by simply rotating the paper-cut figure.



"Non-Parallelograms" **Parallelograms**

Following the intervention plan (see appendix) with minor changes, the idea of continuity was utilized. Nua was asked to use the dynamic parallelogram she constructed by connecting stripes, to create as many parallelograms as possible. After watching a demonstration of the task performed by teacher she was directed to change the shapes **slowly** and stop at each "station" for short period of time. She was also asked to draw each parallelogram she created sequentially and to create as many as possible. When asked *"Are all the shapes you created parallelograms?"* she looked carefully at her sequence of parallelograms, pointed at the rectangle she had drawn and declared it a non-parallelogram, *"Because it has a right angle."*

At this point the teacher suggested “a new agreement”.

Teacher: According to the "old" agreement, parallelograms should not have right angles, but according to the "new" agreement … one should not check angles. So according to our first agreement what is this? And according to our second agreement what is it?

Student: Rectangle and parallelogram (correspondingly).

What remains is the question, “*Which agreement should I follow, the one in which right angles are excluded or the one which accepts right angles?*” The student's choice was not surprising.

Student: I love "rectangle"…

Teacher: In everyday life we take the first agreement, in mathematics we take second agreement. We treat this shape differently in- and outside the mathematics class… I want to convince you that it's neither a crazy nor peculiar idea to choose the second agreement…

[The student's hidden smile suggests her identification with the feeling of "crazy" about naming the rectangle as a parallelogram]

In order to lend support, actually, to “bypass” the logical-mathematical reasoning of classifying a rectangle as a parallelogram, where inclusion plays a crucial role, a vocal and hands gesture demonstration was used to “convince” the student it was natural to consider the shape a parallelogram as well.

Researchers emphasized smoothness and continuity as opposed to discontinuity to be occurred if the rectangle is “taken” out: (1) Using a slow, quiet voice while twisting a striped-made-dynamic-parallelogram held in her hand, the teacher imitated the “noise” of twisting. She asked *"But if we don’t want to include this (the rectangle) as a parallelogram - what happens?"* Repeating the change of shape through the sequence of parallelograms, getting to the point of rectangle, the teacher imitated a car sudden-sharp-stop: *"Stop! Rectangle!"* The use of a “sharp-voiced stop,” when a rectangle is about to be formed produces a feeling of unnaturalness. (2) The teacher also demonstrated her point by running her hand smoothly over her elbow. She suddenly stopped at a certain point and then continued smoothly.

We may add that in an extension of this research which took place later in Israel, we suggested to the students two possibilities: One, in which we actually took the "special shape" (rectangle) off the continuous group of shapes (parallelograms), and the other, when we kept in line also the special shape.

A square among rectangles

Taking off the square

Rectangles without a square

The students chose the second possibility as the one which is more pleasant and "nice" than the one with the "gap" which causes some uneasiness. This strengthens the legitimacy for using the name of the group (e.g. parallelogram) also for their special "members"/shapes (e.g. rectangle).

Teacher: This is why the idea of naming that (the rectangle) a parallelogram is natural and not crazy. Do you agree?

Student: Yes.

Teacher: So now take another look at the shapes you decided were not parallelograms… Try to see if you can create them from the stripes, by changing smoothly the construct of the stripes.

[The student easily decomposed her former choice of "non-parallelogram" and correctly categorized the shapes as parallelograms and non-parallelograms.]

Teacher: Great job! Way to go!

Following the intervention plan (see appendix) with minor changes, when it finally came to its end, Nua could classify correctly all shapes. Moreover, she could point out shapes that she initially had categorized incorrectly. As far as we can judge from her justification, they were not logic-based, level 3 explanations. No inclusion-based decisions or definition-based argumentations were there. Instead Nua checked the characteristics of the shapes to see if they satisfied the "list of requirements" for the chosen name, not only the definition.

In an interview that was held one week after the intervention, Nua told us that *"Before the intervention, I knew the shape to be a rectangle. But after the intervention I recognized the shape to also be a parallelogram."*

Interviewer: …What made you change your mind?

Student: You told me that in mathematics it can be a parallelogram.

Interviewer: But why do you think this shape should be a parallelogram? Why not only a rectangle?

Student: Because two pairs of opposite sides are parallel. So this is a parallelogram.

She described the use of the stripes as effective in changing her former ideas: "*In seeing the shape … seeing the tool… The device is changing… It helped… All these shapes have the properties of parallelograms.*"

DISCUSSION

The above intervention suggests the following process of change.

1. When first asked, along with the rest of her class, to complete a questionnaire that required the classification of *drawn* shapes into parallelograms/non parallelograms, 5 out of 11 answers Nua gave were incorrect. She made incorrect judgment even for prototypes of parallelograms.

At the onset of the intervention, Nua was asked to group *paper-cut* shapes. She was able to correctly judge all prototypes. This may be explained by the evoking of memory on "what a parallelogram" is or by the aid of the visually prepared shapes. Researchers noticed a hesitation when she picked up the "diamond" shape. Finally she decided to place it in the group of "non-parallelograms." The important point to note is that she could not yet judge correctly any of the non-prototype shapes, i.e. those which could not "become" parallelograms by simply rotating the paper-cut figure. For these shapes inclusion relations were needed. These shapes included all right-angle parallelograms.

2. By the end of the intervention, Nua could classify correctly all shapes (see 2nd questionnaire on Table 2). Moreover, she could point out shapes that she initially had categorized wrong.

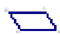
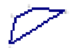
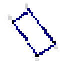
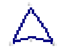
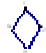
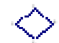
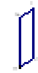
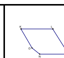
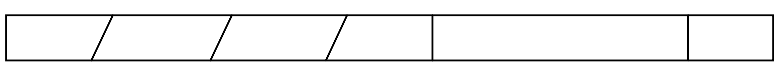
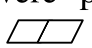
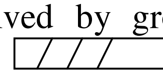
											
1st questionnaire	***No***	No	No	***No***	***No***	No	Yes	Yes	***No***	***No***	No
Grouping task	<u>Yes</u>		No	***No***	***No***		Yes	Yes, ***No***, Yes, ***No***	***No***	***No***, <u>Yes</u>	
2nd questionnaire	Yes	No	No	**<u>Yes</u>**	**<u>Yes</u>**	No	Yes	**<u>Yes</u>**	**<u>Yes</u>**	**<u>Yes</u>**	No

Table 2: Nua's shape classification as parallelogram/non parallelogram (The variety of shapes in the grouping task outnumbered those on the questionnaire; only those that were the same are included here.) Turning points are underlined.

3. In order to obtain more evidence of her ability to identify parallelograms, researchers showed her the following configuration, taken from a Korean mathematics textbook (MEHRD, 2005, p. 77), and asked her to point out all parallelograms.



Nua was able to point out not only the 2 simple figures of prototype parallelograms, but also the "simple" rectangles and parallelograms (including rectangles) that were perceived by grouping the decomposed units of the configuration. e.g.  or 

SUMMARY

Assignments which require the relations of inclusion are essential parts of geometry studies in school and influence students' success in their studies. Junior high school geometry class students in Taiwan (Gal, Lin & Ying, 2006),) in Israel (Hershkowitz, 1989a) and in many other nations including Korea are found to have similar difficulties.

It is believed that the main pitfall in these cases is the gap between the required geometrical thinking-level for such assignments and students' actual level. Unfortunately, acquiring a higher level of thinking is not easy. If the assignments are designed for a higher level, the student might be led into rote learning and relying on memorized definitions. Sometimes these established definitions are incorrect and sometimes students simply pretend to incorporate them in their decision-making. In fact, their answers might be the outcome of their prototypes.

This study looked at the effect of a bypass in which students make decisions using tools that help produce correct results without relying on inclusion or definition.

The tools we offered enlarged students' concept image (Tall & Vinner, 1981) and acted like scaffolding in decision-making. The use of dynamic stripes produced an effect similar to the effect produced by dynamic software (e.g. Arshavsky & Goldenberg, 2005). However, it provided concrete, touchable material which can be used as a mental reference while the student twists, rotates or changes mentally (or even physically) a given shape to determine if it could result from the striped prototype parallelogram. This could tip the scale when a cognitive conflict occurs: "is it or is it not a parallelogram?" Lastly, the controlled use of gestures in teaching (verbal and physical) has already been shown to be a powerful instructional tool (e.g. Goldin-Medow, Kim & Singer, 1999).

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APPENDIX

Is a rectangle a parallelogram? - Intervention plan

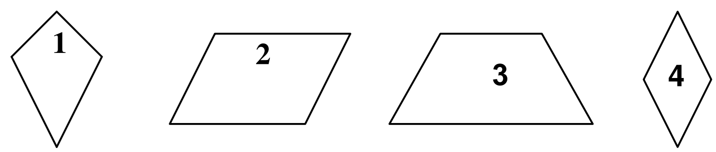
Guideline for the teacher

The intervention is based on van Hiele's phases of learning: inquiry, direct orientation, expliciting, free orientation, integration (e.g. Hoffer, 1983).

A. Inquiry

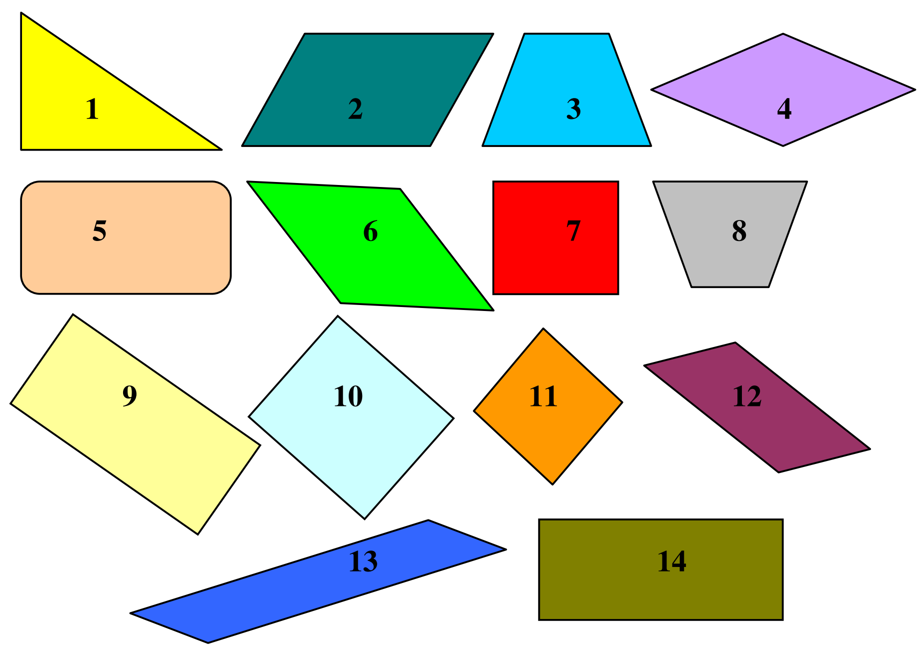
1. A student's concept image of "parallelogram".

Introduce the following shapes, asking: Are there parallelograms among these shapes? If there are any – circle them.



2. Documenting the starting point.

Place the following paper-cut shapes on the table. While their numbers "stand" in front of the student, ask the student to put them into 2 groups: parallelograms and non-parallelograms.



Note: At this stage the teacher should not correct the student if they do not identify special parallelograms (rhombus, rectangle or square). If the student identifies a *non-parallelogram or does not identify prototype, only then should the teacher point at a prototype of a parallelogram.*

3. Reasoning.
   Ask the student to explain why he/ she placed each shape in the category he/ she did.

   Note: At this stage it is essential to trace the student's answer, carefully observing his/her hands, words and gestures, in order to figure out what he means when he uses his/her terminology and concepts.

B. Direct orientation

1. Provide a set of stripes with various lengths, with at least 4 stripes of each length. Ask the student to construct a parallelogram

   Note: Tracing the process of construction, the teacher could watch how the student chooses the stripes: Does he/she compare them side by side (van Hiele level 2) or by trial and error (probably level 1)?

2. Ask the student to use the dynamic parallelogram he constructed in order to create as many parallelograms as possible (without disconnecting the stripes).
   Ask the student to draw several shapes (about 7) from those he went through on transparencies, each shape on a different sheet.

   Note: It is important that the student use the stripes himself.

3. Ask the student to place the drawn parallelograms one after the other, sequentially (according to the inclination, i.e. with growing angles).
4. If the student did not draw a rectangle, draw one yourself and ask the student to put it in the line as well.

   Note: This requirement might be rejected by the student, claiming that this is not a parallelogram. Ask the student to put it in line anyway and to judge if visually speaking the shape is integrated among the others on the continuum.

C. Expliciting

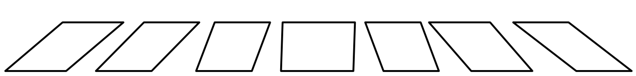
1. Ask the student if all the shapes that he has are parallelograms.

   Note: The teacher may slowly repeat the continuous movement of the stripes and stop in several "stations".

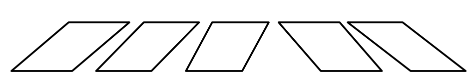
If the student refuses to accept the rectangle as a parallelogram, the teacher should suggest two points of view:

The first is looking at the continuum and where the rectangle's place on the continuum is perceptually natural.

Note: The teacher might suggest taking the rectangle off the continuum and putting it back. Then ask the student to decide from a visual point of view which is more continuous and harmonious.



A sequence of parallelograms with a rectangle



A sequence of parallelograms without a rectangle

The second is looking at the rectangle by itself.

In the first case, it deserves to have the name of the whole group where it is a "legal member", i.e. a parallelogram. But when it stands alone it has the name "rectangle". The teacher will say that in mathematics, as in other cases, we allow two names; therefore we may call it parallelogram as well as rectangle. The teacher will ask if the student can live in peace with this solution.

2. Encourage the idea of having two names for one figure by using the example of the student having his "own" name and being called "9^{th} grader".
   We support it being "legal" and correct to have 2 names, and even to "prefer" one of them. We note that many will probably use the name "rectangle" for that shape. We ask: "But can we also look at it as a parallelogram, if we want to keep continuity of movement and smoothness?"
3. Repeating the smooth movement of the stripes, ask: "Are they all parallelograms?"
4. Now, stop clearly at the "rectangle point" asking: "Is that a parallelogram?"
5. Change the orientation by rotating the shape and repeat those questions.

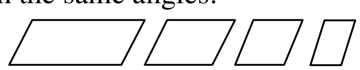
D. Free orientation

Second cycle: Is a rhombus parallelogram?

1. Take 2 (equal) sides off and replace them with two others. Ask the student to repeat the procedure of creating various parallelograms. This time we might skip the drawing. We see that they may all be called parallelograms.
2. Again, we take 2 new (equal) sides off. Replace 2 sides as before, and again show many parallelograms in slow changes.

Note: With this procedure we keep one pair of equal sides for all the parallelograms.

3. Finally only the 4 equal sides remain, and the student repeats the procedure.
4. Now, the teacher re-builds one parallelogram from each pair of sides, where the other pair stays constant to all the parallelograms.
5. Ask the student to order them in a sequence, with the common sides presented "vertically" and all with the same angles:



6. Suggest generalizing the case of a rhombus among the parallelograms, as before.

Third cycle: Is a square a rectangle?

Similar to the previous cycles, produce a sequence of rectangles, where the square is in between. Conclude.

Fourth cycle: Is a square a parallelogram?

Use the sequence of rhombuses that were identified as parallelograms to help the student conclude that the square, which was part of the same sequence, is also a parallelogram.

Use the sequence of rectangles that were identified as parallelograms to help the student conclude that the square, which was part of the same sequence, is also a parallelogram.

E. Integration

1. Discussion: What did you learn about a parallelogram? Give detailed answer.

   How would you tell a friend to construct one?

   What are its properties?

   Can all angles be equal? Construct this shape. Do you know another name for such a shape?

   Can all sides be equal? Construct this shape. Do you know another name for such a shape?

   Can all sides and angles be equal? Construct this shape. Do you know another name for such a shape?

   Discuss the idea of continuum and of having two or more names in all of these cases.

2. Re-provide the first collection of shapes and group them into parallelograms and non-parallelograms.
3. Re-provide the first collection of shapes and group them into rectangles and non-rectangles.

   If correct, also ask the student to give more than one name to some of the shapes.

4. Ask if the student to point out where he was mistaken in the beginning of the intervention.