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Pastoral Calculus Deconstructed

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Calculus becomes pastoral calculus killing the interest of the student by presenting limit- and function- based calculus as a choice suppressing its natural alternatives. Anti-pastoral sophist research searching for alternatives to choice presented as nature uncovers the natural alternatives by bringing calculus back to its roots, adding and splitting stacks and per-numbers.

The background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$'.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

Ancient Greece saw a fierce controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronising rulers.

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronisation with democracy. Two democracies were installed: one in US, and one in France, now having its fifth republic.

In France, sophist warning is kept alive in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronising categories, discourses and institutions presenting their choices as nature (Tarp 2004). Derrida recommends that pastoral categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to pastoral discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimising their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus being able itself to falsify its number but not its word, which makes numbers nature and words choices, becoming pastoral if suppressing their alternatives; meaning that a thing behind a word only shows part of its nature through a word, needing deconstruction to show other parts.

Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. The sticks can be rearranged to an eight-icon 8 containing the eight sticks, written as 8. The sticks can be collected to one eight-bundle, written as 1 8s. The sticks can be ‘decimal-counted’ in 5s by bundling & stacking, bundling the sticks in 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, cup-writing gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number:

8 = 1 5s + 3 1s = 1)3) = 1.3 5s

So the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size.

Since ten is chosen as a standard bundle-size, no icon for ten exists making ten a very special number having its own name but not its own icon. This has big technical advantages as shown when comparing the Arabic numbers with the Roman numbers, that has a special ten-icon X, but where multiplication as XXXIV times DXXVIX is almost impossible to do. But without its own icon ten creates learning problems if introduced to early. So to avoid installing ten as a cognitive bomb in young brains, the core of mathematics should be introduced by using 1digit numbers alone (Zybartas et al 2005).

Also, together with choosing ten as the standard-bundle size, another choice is made, to leave out the unit of the stack thus transferring the stack-number 2.3 tens to what is called a natural number 23, but which is instead a choice becoming pastoral by suppressing its alternatives. Leaving out units might create 'mathematism' (Tarp 2004) true in the library where 2+3=is true, but not in the laboratory where countless counterexamples exist: 2weeks+3days = 17days, 2m+3cm = 203cm etc.

The Nature of Operations

Operations are icons describing the process of counting by bundling & stacking.

The division-icon '/2' means 'take away 2s', i.e. a written report of the physical activity of taking away 2s when counting in 2s, e.g. 8/2 = 4. The multiplication-icon '4*' means '(stacked) 4 times', i.e. a written report of the physical activity of stacking 2-bundles 4 times, T = 4*2

Subtraction '- 2' means 'take away 2', i.e. a written report of the physical activity of taking away the bundles to see what rests as unbundled singles, e.g. R = 9 – 4*2. And addition '+2' means 'plus 2', i.e. a written report of the physical activity of adding 2 singles to the stack of bundles either as singles or as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the full process of 're-counting' or 're-bundling' 8 1s in 5s can be described by a 'recount or rebundle formula' containing three operations, together with a 'rest formula' finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = 8 – 1*5 = 3.

All the recount formula T = (T/b)*b says is: the total T is first counted in bs, then stacked in bs.

This recount formula cannot be used with ten as the bundle-size since we cannot ask a calculator to calculate T = (8/ten)*ten. However, this is no problem since the moment ten is chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the recounting formula but by simple multiplication. To re-bundle 3 8s in tens, instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24.

The Nature of Formulas

Using the recount formula, the counting result can be partly predicted on a calculator where 9/4 = 2.something. This predicts that recounting 9 in 4s will result in 2 4-bundles and some singles. The number of singles can be predicted by the rest formula R = 9 – 2*4 = 1. So (9/4)*4 is 2.1 4s.

Thus the calculator becomes a number-predictor using calculation for predictions. This shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the 'laboratory'. Historically, this enabled mathematics to replace pastoral belief with prediction, and to become the language of the natural sciences.

The Nature of Equations

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are re-bundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be re-bundled to 7 1s. Obviously, we must take the 3 singles away from the 7 1s to get the unknown bundle-size: x = 7 – 3. So technically, moving a number to the other side changing its calculation sign solves this equation: If x + 3 = 7, then x = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are re-bundled to 7 1s. The equation x * 3 = 7 describes the reversed bundling asking how 7 1s can be re-bundled to 3s. Using the re-bundling procedure and formula, the answer is T = 7 = (7/3)*3, i.e., x = 7/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 = 7, then x = 7/3.

The statement 2 * 3 + 1 = 7 describes a bundling where 2 3-bundles and 1 single are re-bundled in 1s. The equation x * 3 + 1 = 7 describes the reversed bundling asking how 7 1s can be re-bundled in 3s leaving 1 unbundled. Obviously, we first take the single unbundled away, 7 – 1, and then bundle the rest in 3s, giving the result x = (7-1)/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 + 1 = 7, then x = (7-1)/3.

The statement 2 * 3 + 4 * 5 = 4.2 * 6 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in 6s. The equation 2 * 3 + x * 5 = 4.2 * 6 describes the reversed bundling asking how 4.2 6s can be re-bundled to two stacks, 2 3s and some 5s. Obviously, we first take the 2 3s away, and then bundle the rest in 5s. Again technically, moving a number to the other side changing its calculation sign solves this equation: If 2 * 3 + x * 5 = 4.2 * 6, then x = (4.2 * 6 – 2 * 3)/5

The Nature of Calculus

The statement 2 * 3 + 4 * 5 = 3.2 * 8 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in the united bundle-size 8s. This is 1digit integration. The equation 2 * 3 + x * 5 = 3.2 * 8 describes the reversed bundling asking how 3.2 8s can be re-bundled to two stacks, 2 3s and some 5s. This is a 1digit differential equation solved by performing 1digit differentiation:

If 2 * 3 + x * 5 = 4.2 * 6, then x = (4.2 * 6 – 2*3)/5 = (T – T1)/5 = ΔT/5

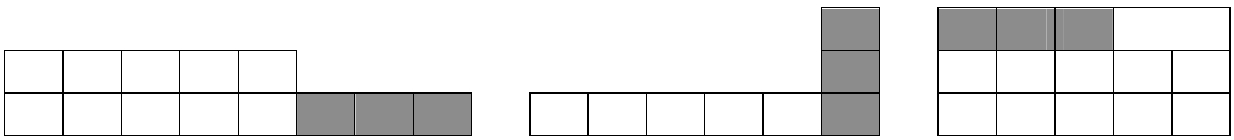
The Nature of Fractions

Once ten has been chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the re-counting formula but by simple multiplication. To re-bundle 3 8s in tens, instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24. Now tables are practiced for re-bundling 2s, 3s, 4s etc. in tens.

With multiplication taking over there is no more need for division in re-bundling and recounting. So division takes on a new meaning in 'per-numbers': If 2 kg costs 8 $, then the unit-price is 8$ per 2kg, i.e. 8$/2kg = 8/2 $/kg. Thus if 4kg cost 5$, the guide-equation '4kg = 5$' is used when re-counting the actual kg-number in 4s, and re-counting the actual $-number in 5s:

10kg = (10/4)*4kg = (10/4)* 5$ = 12.5$, and 18$ = (18/5)*5$ =(18/5)*4 kg = 14.4kg.

Division is also part of fractions, originally occurring if, instead of placing 3 singles besides the existing stack of 5-bundles, the 3 singles are bundled as a 5-bundle and put on top of the 5-stack giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.



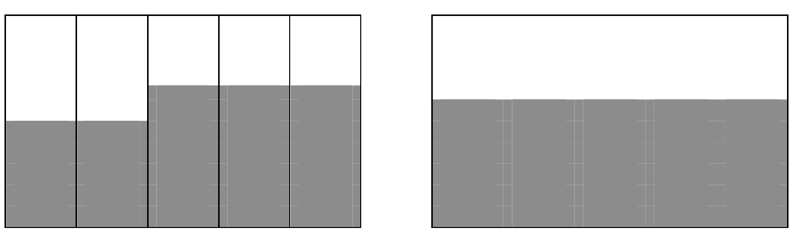
T = 2*5 + 1*3 = 2.3 *5 = 2 3/5 * 5

When adding fractions it is important to reintroduce the units to avoid scaring the learners with mathematism as when performing the following 'fraction test' the first day of secondary school:

The teacher:	The students:
Welcome to secondary School! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. So to add numbers the units must be included, also when adding fractions. And adding fractions f is basically integration:

T = 1/2 *2 + 2/3 *3 = Σ (f*Δx) later to become $\int$ f dx



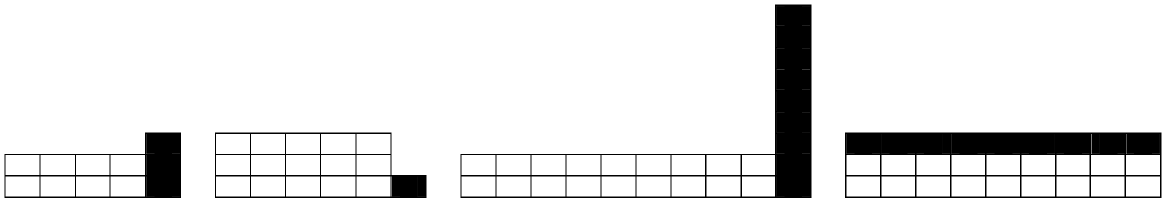
T = 1/2 *2 + 2/3 *3 = 3/5 *5

Primary School Calculus

In primary school integration means integrating two stacks into one where the bundle-size is the sum of the stacks' bundle-sizes. Thus a typical integration problem is 2.3 4s + 3.1 5s = ? 9s.

Manually, integration means placing the two bundle-stacks next to each other; then placing the two 1-stacks on top of each other; then moving any uncompleted bundle to the 1-stack; and finally move any bundles from the 1-stack to the bundle-stack. cup-writing reports this process:

2.3 4s + 3.1 5s = 2)3) + 3)1) = 2)3) + 2)1+1*5) = 2)3+6) = 2)9) = 2+1)0) = 3)0) = 3.0 9s

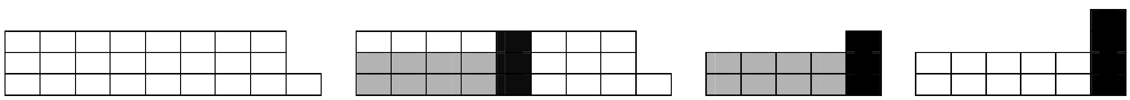


2.3 4s + 3.1 5s = 2.9 9s = 3.0 9s

Reversing the integration of two stacks becomes differentiation. Thus a typical differentiation question is 2.3 4s + ? 5s = 3.1 8s

Manually, differentiation means taking away a stack of bundles and a stack of 1s from a bigger stack; and then recount the rest in the given bundle size. Removing stacks reports this process:

? = (3.1 8s - 2.3 4s)/5 * 5 , later to be written as a differential quotient $(T – T1)/5 = \Delta T/5$.



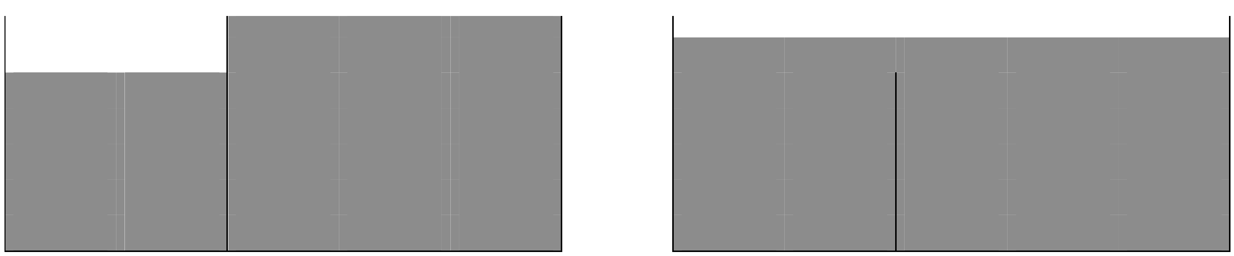
3.1 8s = 2.3 4s + ? 5s = 2.3 4s + 2.4 5s

Middle School Calculus

In middle school integration means integrating two fractions or per-numbers into one. Thus a typical integration problem is 1/2 of 2 + 2/3 of 3 = ? of 5; and 10% of 2 + 40% of 3 = ? of 5; and 2kg at 6$/kg + 3kg at 9$/kg = 5kg at ? $/kg.

Manually, integration means drawing next to each other two rectangular pools and then finding the average water-level if the separating wall is removed. Note-writing reports this process:

2 kg at 6 \$/kg = 2*6 = 12 \$
3 kg at 9 \$/kg = 3*9 = 27 \$
5 kg at ? x/kg = 5*x = 39 \$, so x = 39/5 = 7.8 \$/kg



1/2 of 2 + 2/3 of 3 = ? of 5

Reversing the integration of two pools becomes differentiation. Thus a typical differentiation question is 2kg at 5$/kg + 3kg at ?$/kg = 5kg at 6 $/kg.

Manually, differentiation means drawing two rectangular pools next to each other and then finding the resulting water-level of the right pool after water is pumped from the left pool.

Mentally, combined subtraction and division reports the manual process:

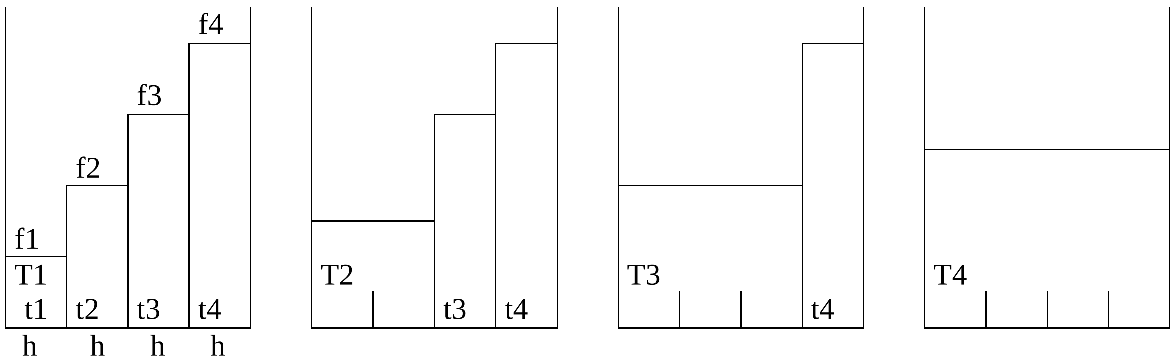
$2*5 + 3*x = 5*6$, $x = (5*6 – 2*5)/3$, later to be written as $(T – T1)/5 = \Delta T/5$.

High School Calculus

In high school, integration means integrating many per-numbers into one. Thus a typical integration problem is: 7 seconds at 2 m/s increasing to 4 m/s totals 7 seconds at ? m/s in average.

Manually, integration means drawing next to each other many rectangular micro-pools with width h and water level described by a formula f; and then finding the formula describing the water level when the walls are removed one by one.

Mentally, cumulating describes the manual process: The volume of pool1 is $t1 = f1*h$. The total volume then is $T4 = T3 + t4 = t1 + t2 + t3 + t4 = \Sigma\ ti = \Sigma\ fi * h$

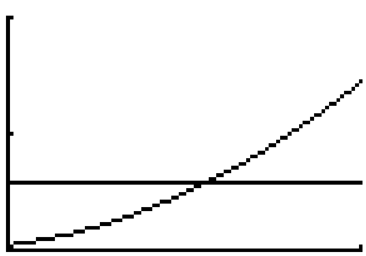


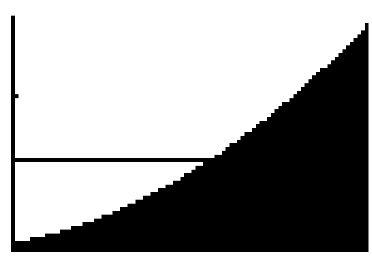
Reversing the integration of pools becomes differentiation. Thus a typical differentiation question is: 5 seconds at 1.82 m/s in average + 0.1 second at ? m/s totals 5.1 seconds at 1.84 m/s in average.

Manually differentiation means drawing two rectangular pools next to each other and then finding the resulting water-level of the right pool after water is pumped from the left pool.

Mentally combined subtraction and division reports the manual process:

$T3 + f4*h = T4$, $f4 = (T4 – T3)/h$, now to be written as $\Delta T/h = dT/dx$ if $h = dx$ is mikro-small.





In the case of many micro-pools a Graphical Display Calculator (GDC) can do the drawing.

The Choices of Pastoral Calculus

Modern calculus still believes in the existence of sets despite of Russell's paradox. Instead of introducing calculus as integrating per-numbers in middle school it waits for algebra to define the real numbers. Then the concept of a limit can be given its ε–δ definition by calculus believing this gives a precise meaning to the term 'mikro-small'; but forgetting that δ is the mikro-number giving the level of exactness described by ε.

So instead of working with mikro-numbers, modern calculus presents both the derivative and the integral as examples of the concept limit, which creates big problems to learners. Also the so-called fundamental theorem of calculus by it naming alone is presented as a deep insight where instead it is a mere banality as shown below. .

The Natural Alternatives of Anti-pastoral Calculus

To uncover natural alternatives to the choices of modern calculus, becoming pastoral by suppressing its alternatives, anti-pastoral sophist research sees calculus as a natural science exploring the natural fact multiplicity rearranged by bundling and stacking. In this approach the roots of calculus is found to be adding two stacks in a combined bundle-size, as shown above. Later calculus also applies when adding per-numbers as $/kg or m/s. Here integration means predicting the area under the per-number graph to predict the total $-number or m-number; and differentiation means predicting the gradient on the total $-graph or m-graph to predict the per-number.

As to finding gradient formulas given a total formula, and finding area-formulas given a per-number formula, the method of natural science can be used involving observation, induction and testing predictions.

First the GDC's gradient calculator dy/dx is validated if its predictions are verified and never falsified on examples with known gradient. Thus no matter where the dy/dx-number is calculated on the graph y = 2.7 x + 4, the answer is the expected dy/dx = 2.7.

Now a table can be set up for the relation between x and the gradient-number dy/dx on the graph y = x^2. Using regression, it turns out that a gradient-formula dy/dx = 2x can be induced and used for deducing predictions that all are verified. In this way the experimental method of natural science can be used to find the different gradient-formulas.

Next the GDC's area calculator ∫f(x)dx is validated if its predictions are verifying and never falsified on examples with known area. Thus no matter where the area-number is calculated on the graph y = 2, the answer agrees with the known formula A = 2*(x2-x1).

Now a table can be set up for the relation between x and the area-number under the graph y = x^2 from 0 to x. Using regression, it turns out that an area-formula A = x^3/3 can be induced and used for deducing predictions that are all verified. In this way the experimental method of natural science can be used to find the different area-formulas.

As to finding the area-number using the fundamental theorem of Calculus, as simple observation shows that the following statement cannot be falsified:

The total change = the cumulated step-change = y end – y start, or $\Delta y = \Sigma\Delta y = y2 - y1$.

This statement does not depend upon the size or number of changes, so it also applies for small mikro-changes:

$\Delta y = \int dy = y2 - y1$, or $\Delta y = \int y' \, dx = y2 - y1$ if dy is re-counted in dxs as $dy = dy/dx * dx$.

So when calculating the area under an f-graph, if f can be re-written as a change-formula $f(x) = dy/dx$, then the area $\int f(x)\, dx$ can be written as $= \int dy$ and calculated as the difference y2 - y1.

Number y	Step-change Δy	Cumulated step-change $\Sigma\Delta y$	Total change $\Delta y = y2 - y1$
2			
5	3	3	3
4	-1	2	2
9	5	7	7

Conclusion

Modern mathematics finds it natural to postpone calculus until the end of high school or the beginning of university, wanting to present it as metamatics, i.e. as an example of the higher abstractions as sets, functions, real numbers and limits; in spite of the fact that historically calculus was developed before these abstractions. However, this choice turns out to be a pastoral choice suppressing its natural alternatives uncovered by anti-pastoral sophist research searching for alternatives to choice presented as nature. The natural alternative is to introduce calculus in primary school as adding stacks in united bundle-size, and to reintroduce calculus in middle school as adding per-number and fractions with units, and finally using the methods of natural science to make high school calculus limit-free.

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