PROCEDURAL EMBODIMENT AND QUADRATIC EQUATIONS

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In this paper, we present the results of a research study involving 77 14-15 year-old students working with quadratic equations. Data were analysed in the light of a theoretical framework that considers three different worlds of mathematics (Tall, 2004; Lima, 2007). Evidence shows that students give to equations and the solving methods they use meanings related to procedural embodiments (Lima & Tall, 2008), taking symbols as physical entities that can be moved around, "putting them on the other side" of the equation, with the additional magic of, for example, "changing signs", or "transforming the exponent in a square root" in the case of quadratics. Procedural embodiments may be effective in some cases, but it is necessary to relate them to their underlying mathematical concepts, in order to make it possible for students to relate the meanings they give to equations to the algebraic principles they should be connected.

INTRODUCTION

> "*When I saw the teacher coming into the room shouting 'get out your calculators' and writing 'Equation' on the board, the whole class was silenced and I thought 'My God, what is this?' Then, she started to put signs, brackets, and another abstract codes. I was in panic, no exaggeration; all feelings from mathematics came through. When she saw that I wasn't doing the book exercises, she shouted 'Fabioooooo get your book and find the solutions for the problems'. I thought that I should stop, but she started to talk about the various qualities a student should have, like dedication and endeavour. I was scared. Every time I heard her say 'quadratic formula', 'tangent', 'cosine', my mind was full of doubts; while she was talking about the unknown, I had a knot in my head. So, I've made an exception and thought 'it is better to stop here'. I've remembered my reasoning capacity, got the pencil, the rubber and other instruments and ways of work, and started to do the exercises.*"
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> A 15 year-old student's statement on his first impression about equation.

This statement shows how frightening Algebra can be for students, a reason why, for many years, there have been a large amount of research studies in Mathematics Education concerning its teaching and learning and, specifically, a search for reasons why students face so many problems in learning equations and methods for solving them. Some of these research studies diagnose student's mistakes when solving equations (Sleeman, 1984; Payne & Squibb, 1990; Freitas, 2002); others, discuss the understanding students have about an equation (Dreyfus & Hoch, 2004). Apparently, students from many different countries make the same mistakes. Reasons why such mistakes appear to be related to the misinterpretation, by students, of techniques to solve equations and the lack of meaning attributed to the mathematical symbols used (Linchevski & Sfard, 1991; Cortés & Kavafian, 1999).

Attempts to minimise such students' difficulties have involved the use of concrete models, like the balance or the geometric models (Vlassis, 2002; Filloy & Rojano, 1989). These have been shown to be very effective in helping students to understand the equality between the two sides of an equation, but they do not support situations in which negative or non-integer numbers are involved.

It is our belief that it is not simply a misinterpretation of techniques or a lack of meaning for equations that prompt students to use mal-rules (Sleeman, 1984; Payne & Squibb, 1990). Students do give meanings to the ways of working they create (Lima & Tall, 2008) and in order to understand why students make mistakes in solving equations, it is necessary to search for the roots of these meanings. In this search, we have developed a research study that involved 77 14-15 year-old students and four instruments of data collection, with questions in which students should solve linear and quadratic equations and problems, write about equations, solving methods, formulas and the mathematical concepts behind them.

Analysis of collected data was made in the light of a theoretical framework, still in development, that considers three different worlds of mathematics, *conceptual embodied world*, *proceptual symbolic world*, and *formal axiomatic world*. This theoretical framework made it possible to analyse students' work considering not only the use of symbols, as with process-object theories, or embodied aspects, as in embodied cognition theories, but the relation of both these aspects, along with formal characteristics of equations and solving methods. Our research has shown that the meanings student give to their work with equations is not always related to the mathematical concepts, as we might wish, but are based on *procedural embodiments* (Lima & Tall, 2008), in which symbols are seen as physical entities that are moved around, carrying an additional *magic* that changes something in the symbol or number when it is moved. In this paper, we discuss procedural embodiments in quadratic equations found in our research studies and their influence on students' successes and mistakes.

THREE WORLDS OF MATHEMATICS

Neither process-object not embodied cognition theories were suitable to analyse data collected from our research because they emphasise a single characteristic of mathematical concepts, the role of symbols and embodied aspects respectively. Rather, we have used a theoretical framework of long term learning, based on the understanding that there are at least three different kinds of mathematical concepts (Gray & Tall, 2001), based on the human activities of perception, action and reflection, that inhabit three different worlds of mathematics (Tall, 2004; Lima, 2007; Lima & Tall, 2008):

A *conceptual embodied* world of perceptions, in which an object and its properties are observed, in a way that an individual can make sense of, and verbally describe.

A *proceptual symbolic* world, in which mathematical entities are symbolised, and procedures are use to perform actions upon them by using procedures that may be flexibly seen as both the procedure and the product of this procedure, the concept, in the duality of *procepts* (Gray & Tall, 1994).

A *formal axiomatic* world of axioms, properties, definitions and theorems, with which it is possible to construct the body of Mathematics.

Quadratic equations and different Worlds of Mathematics

In the case of quadratic equations, the geometric model can be used as an embodiment for such equations, although it is difficult to give sense to the balance model. The embodied world would be also at play when dealing with equations in which the operations can be undone, such as $a(x-u)^2+v=0$.

The quadratic formula (known in Brasil as Bhaskara's formula) represents a symbolic way of solving equations. It is used specially for equations in the form $ax^2+bx+c=0$. Equations of the form $a(x-x_1)\cdot(x-x_2)=0$ can also be solved by the formula. Alternatively, it is possible to use a formal characteristic of real numbers which states that if a product is zero, one of the factors must be zero. In this case, we believe that both symbolic and formal worlds are at play, even if not at all their potentialities.

Working with different procedures to solve quadratic equations, students can develop a flexible way of dealing with them, understanding that the main important aspect is the *effect* resulted from the use of any procedure, and not a single procedure itself.

THE RESEARCH STUDY

The study presented in this paper is part of our doctoral research, developed at PUC/SP (Brazil) and at the University of Warwick (UK). We worked with three groups of high school students, one of 32 14-year-olds, one of 28 15-year-olds, both from a public school in Guarulhos/SP; and one group of 20 15-year-olds from a private school in São Paulo/SP. We conducted three 100 minutes sessions, in which the class teacher has administered our instruments to collect data: a conceptual map, a questionnaire and an equation solving task. After an initial analysis of data, 15 of the students were interviewed by the researcher in the presence of an observer. Interviews were tape recorded for further analysis. In this paper, we focus on the procedural embodiments identified in the students' work with quadratic equations.

Quadratic equations in the instruments

In both questionnaire and equation solving task, students were asked to solve quadratic equations. There were two in the questionnaire: $t^2 - 2t = 0$ and $(y-3)\cdot(y-2)=0$, and four in the equation solving task: $3l^2 - l = 0$; $r^2 - r = 2$; $a^2 - 2a - 3 = 0$ and $m^2 = 9$. We also had a question in the questionnaire, in which the students are asked to analyse and comment the solving method for the equation $(x-3)\cdot(x-2)=0$ given by "John":

> To solve the equation $(x-3)\cdot(x-2)=0$ for real numbers, John answered in a single line that:
>
> "$x = 3$ or $x = 2$"
>
> Is his answer correct? Analyse and comment John's answer.

Figure 1: Question 8 of the questionnaire.

We have analysed the solving methods students use, and how they justify the validity of such methods. Specifically for question in Figure 1, we have searched for mathematical reasons to accept or refuse John's solution. In all questions, we have looked for procedural embodiments students may have used.

RESULTS AND ANALYSIS

From these instruments, we have found that students understand an equation just like a calculation, such as addition or multiplication in the embodied world, with integer numbers. In this way, the unknown is not on focus as an essential characteristic of an equation, and the equals sign is seen as an operation sign (Kieran, 1981).

The symbolic world is in focus because the students use techniques to solve equation, procedures to perform operations with symbols. However, our evidence indicates that students are not aware of the duality of the symbols as processes and concepts. What they do is to analyse what they have at hands in each step of the solving process, searching for which "magic" is suitable at what moment: to change signs or to put the symbol underneath what is there. As algebraic principles are detached from this procedure, they may use any of these magic procedures with no criteria, which prompt them to create mal-rules and ways of working.

From our research, it is possible to conclude that those students do give meaning to equations and solving methods. However, these are not necessarily related to mathematical principles anymore, but have to do with the actions students carry out when solving equations. Instead of "doing the same operation to both sides of the equation", students are "passing a term to the other side", with the additional *magic* of "changing signs" or "putting it underneath what is there". Therefore, meaning becomes related to the movement of symbols and magic. Algebraic symbols seem to be physical entities that can be moved around at will. As the reasons for why the technique of transposing terms are not clear for the students, they accept without much questioning that some kind of magic is necessary to complete the physical movements. We call *procedural embodiment* (Lima & Tall, 2008) this movement of symbols with the additional magic that must be performed to get the right technique.

Solving quadratic equations

The quadratic formula was the only mathematically valid method to solve quadratics that this research study's subjects knew. Unfortunately, it was not always a successful one. No more than *seven* students got at least one of the six quadratic equations of all instruments right. Apparently, the symbolic world is the one emphasised in their work; however, there was little evidence of the flexibility necessary to choose a procedure appropriate for each situation, not surprisingly perhaps as their teachers confirmed that classroom practice was based entirely on the formula. This meant many students used the formula to solve $(y-3)\cdot(y-2)=0$ or $3l^2 - l = 0$. Three students used the formula even for the equation $m^2 = 9$, although only one found the right roots.

Another way of solving quadratics attempted by the students who participated in our study was "transforming" a quadratic equation into a linear one. This was done in various forms. For instance, nine students simply replaced m^2, r^2 or a^2 respectively for *m*, *r* and *a*, and then solved the equation as if it were a linear one. Others apply the exponent associated to the unknown with its coefficient instead (Figure 2), and nine students believe that m^2 is the same as *m* and *m* which is, for them, equals to $2m$ (Figure 3).

We also observed the use of procedural embodiments in students' work with quadratics. Sometimes such embodiment is associated with the exponent of the unknown. To solve equation $m^2 = 9$, students do as shown in Figure 4. The explanation for the procedure performed in the second line is given by a student in interview as: *"The power two passes to the other side as square root"*. In this explanation, the student makes it clear that there is a movement of the exponent (passes to the other side) and an additional magic of being transformed in a square root. Although such procedural embodiment results in one of the equation's roots, it lacks completely mathematical meaning. In the interview, this and other students do not perceive the absence of one of the equation's roots. We believe that their work with linear equations may have influenced them to be satisfied with only one solution.

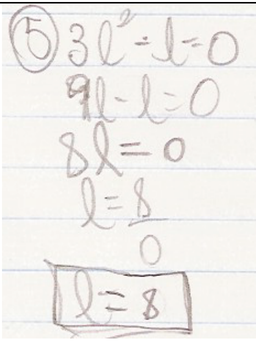

Figure 2: Using the power on the coefficient

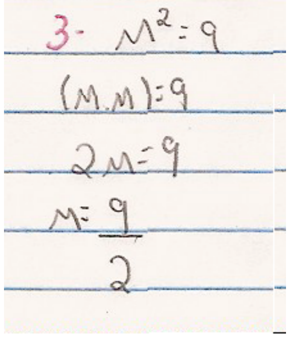

Figure 3: m^2 taken as the same as $2m$

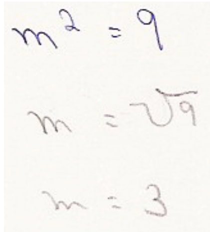

Figure 4: Procedural embodiment in quadratics

For Question 8 (Figure 1), we found that 30 students out of 77 claimed that John's solution is right. However, none of them said that it was because one of the factors of the multiplication should be zero. One of the several reasons they give to justify the solution is that "he must have used quadratic formula in his mind", showing how strong is the use of the formula for these students. One student who uses the formula incorrectly and gets different results from John's, even says that *"I don't know, but I think that John is wrong and I think that my way is right; I said my way, not my results, ok?"*.

In this question, a procedural embodiment is used in a way that brings success, although by only a few students (three in the questionnaire and one during interview). One of them says that John is right "because putting $x = 3$ or $x = 2$, it gives the number zero", while other two actually replaces the values in the equation (Figure 5).

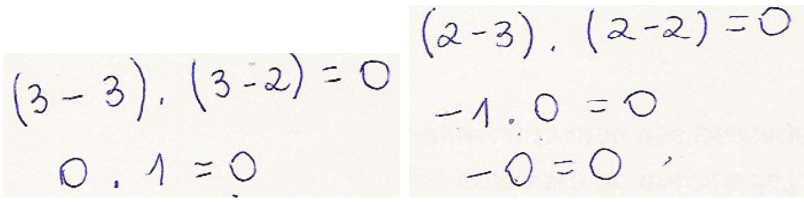

Figure 5: A student replaces the values of x in the equation.

The following discussion on interview shows one of these students' thoughts:

Student: To see if the answer is right, I should have put 3 here [in the place of *x*] and seen what result I would get, and then another calculation with 2.

Interviewer: Why have you put 3 in the place of *x*, and then 2 in the place of *x*?

Student: Because here it says that *x* is equal to 3 so, if *x* is 3, then I replace the number to see what I get.

Interviewer: And what happens if the result is the same as the one in the equation?

Student: If it is zero, then x is 3.

We consider his reasoning as a procedural embodiment because he explicitly *puts* the numbers in the place of x in the equation, moving the symbols and replacing them. In this case, such embodiment resulted in a good answer for the question. We believe that, if a procedural embodiment is used combined with mathematical meaning, it may bring good results for the students.

FINAL DISCUSSION

In this paper, we have presented the use of a theoretical framework of long term learning that made it possible to find out that students do give meaning to equations. The problem is that such meanings are not related to mathematical concepts, as we wish. They derive from the procedures students use to solve equations, specially the movement of symbols from one side to the other. In addition, as students seem not to be aware of algebraic principles behind such movement, they have to use a "touch of magic" to finish using the procedure. Quadratic equations involve a different wave of the wand, adding the new magic of passing the exponent of the unknown to the other side as a square root, while a non-familiar situation. Procedural embodiments however are not always associated with failure, the unfamiliar situation presented Figure 1 enable a few students to develop an appropriate way of moving symbols around.

The teachers with whom we worked throughout the project (Lima, 2007) believed that teaching the quadratic formula (and the techniques for linear equations) would guarantee success for their students; as such methods may be used to solve any kind of equation. Instead, students were confined to a single way out, and prevented from developing a flexible way of solving different equations and a proceptual thinking (Gray & Tall, 1994). In addition, formulas and techniques ended up being detached from algebraic principles, leading students to believe that a symbol is a physical entity and, as such, can be freely treated, subject to any kind of "rule". Of course, the "proper" use of procedural embodiments can be useful and bring success for students. However, in order to prevent them from creating mal-rules (Sleeman, 1984) or ways of working (Lima & Tall, 2008); it is necessary to make connections with characteristics of the formal world as well. Being able to choose a suitable method for each situation is not a matter of memorizing them all. The main point is to understand why a method is suitable for the situation in hand. In this case, we believe that characteristics of formal world may help.

For further research, we suggest the study of quadratic equations, not by studying only quadratic formula, but various procedures to solve different kinds of quadratics, in order to proportionate the development of a proceptual thinking. In following such a path, we may also find out whether there are different kinds of procedural embodiments, as well as the effects they bring for learning.

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