PASTORAL ALGEBRA DECONSTRUCTED

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Presenting its choices as nature makes modern algebra pastoral, suppressing its natural alternatives. Seeing algebra as pattern seeking violates the original Arabic meaning, reuniting. Insisting that fractions can be added and equations solved in only one way violates the natural way of adding fractions and solving equations. Anti-pastoral grounded research identifying alternatives to choices presented as nature uncovers the natural alternatives by bringing algebra back to its roots, describing the nature of rearranging multiplicity through bundling & stacking.

The Background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs the outside world no more and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: M = {A□A∉A)}. Statement: M∈M ⇔ M∉M'. Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

In ancient Greece a fierce debate took place between two different forms of knowledge represented by the sophists and the philosophers. The sophists argued that to protect democracy peopled needed to be enlightened to tell choice from nature in order to prevent the emergence of patronization presenting its choices as nature. The philosophers argued that democracy should be abolished since everything physical are examples of meta-physical forms only visible to philosophers educated at Plato's academy, and who then should become the patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple follows its own will, people could do the same and replace patronization with democracy. Two democracies were installed: one in the US, and one in France, now having its fifth republic.

In France, sophist skepticism is kept alive in the poststructuralist thinking of Derrida, Lyotard and Foucault warning against pastoral patronizing categories, discourses and institutions presenting their choices as nature (Tarp, 2004). Derrida recommends that pastoral categories be 'deconstructed'. Lyotard recommends the use of postmodern 'paralogy' research to invent alternatives to pastoral discourses. And Foucault uses the term 'pastoral power' to warn against institutions legitimizing their patronization with reference to 'scientific' categories and discourses.

On the basis of the ancient and the contemporary sophist skepticism, a research paradigm can be created called 'anti-pastoral sophist research' deconstructing pastoral choices presented as nature by uncovering hidden alternatives. Thus anti-pastoral sophist research doesn't refer to but deconstruct existing research by asking 'in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?'

To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the American enlightenment democracy, resonating with Piaget's principles of natural learning (Piaget, 1970).

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. They can be rearranged to an eight-icon containing the eight sticks, written as 8. They can be collected to one eight-bundle, written as 1 8s or 1*8. And they can be counted by bundling & stacking, bundling the sticks in e.g. 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, 'cup-writing' gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number:

8 = 1 5s + 3 1s = 1)3) = 1.3 5s

So the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size.

The bundle-size icon, e.g. 5, is not used in decimal-counting as shown by the oral and written sequence: One, two, three, four, bundle, bundle1, bundle2, bundle3, bundle4, 2 bundle, 2bundle1 etc. written as 0.1, 0.2, 0.3, 0.4, 1.0, 1.1, 1.2, 1.3, 1.4, 2.0, 2.1.

Choosing ten as bundle-size, it needs no icon making ten a special number having its own name but not its own icon. The advantage of decimal-counting is seen when comparing decimal numbers with the Roman numbers, that has a special ten-icon X, but where multiplication as XXXIV times DXXVIX is almost impossible to do. Without its own icon, however, ten creates learning problems if introduced to early. So to avoid installing ten as a cognitive bomb in young brains, the core of mathematics can be introduced by using 1digit numbers alone (Zybartas et al, 2005).

Also, together with choosing ten as bundle-size, another choice is made, to leave out the unit of the stack thus transferring the stack-number 2.3 tens to what is called a natural number 23, but which is instead a choice becoming pastoral by suppressing its alternatives. Later leaving out units create 'mathematism' true in the library where 2+3 = 5 can be proven true, but not in the laboratory where countless counterexamples exist: 2weeks + 3days = 17days, 2m + 3cm = 203cm etc.

The Nature of Operations

Operations emerge as icons describing the process of counting by bundling & stacking. Saying 'the total is the total bundled in bundles and stacked in bundles' can be written in an abbreviated form as T = (T/b)*b, the recount-formula. In this way letters are introduced not as numbers, but as abbreviations of words. And here letters are replaced with numbers instead of numbers being replaced by letters, as is the tradition in algebra.

The division-icon '/2' means 'take away 2s', i.e. a written report of the physical activity of taking away 2s when counting in 2s, e.g. 8/2 = 4. The multiplication-icon '4*' means '(stacked) 4 times', i.e. a written report of the physical activity of stacking a 2-bundle 4 times, e.g. T = 4*2

Subtraction '- 2' means 'take away 2', i.e. a written report of the physical activity of taking away the stack to see what rests as unbundled singles, e.g. R = 9 – 4*2, the rest-formula. And addition '+2' means 'plus 2', i.e. a written report of the physical activity of adding 2 singles to the stack of bundles as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the result of re-counting 8 1s in 5s can be predicted by two formulas: the recount-formula finds the number to the left of the decimal point, and the rest-formula finds the number to the right:

T = 8 = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = T – 1*5 = 3.

With ten as bundle-size, recounting becomes multiplication: to recount 3 8s in tens, instead of writing T = (3*8)/10 * 10 = 2.4 * 10, we simply write T = 3*8 = 24. Now tables are practiced for rebundling 2s, 3s, 4s etc. in tens. Thus the root of multiplication is division, taking away tens.

The Nature of Equations

The root of equations is the fact that any operation and calculation can be reversed.

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are rebundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be rebundled to 7 1s. Obviously, we must take the 3 singles away from the 7 1s to get the unknown bundle-size: x = 7 – 3. So moving a number to the other side of the equation sign, by changing its calculation sign solves this equation: If x + 3 = 7, then x = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are rebundled to 7 1s. The equation x * 3 = 7 describes the reversed bundling asking how 7 1s can be rebundled to 3s. Using the rebundling procedure and formula, the answer is T = 7 = (7/3)*3, i.e. x = 7/3. Again, moving a number to the other side changing its calculation sign solves this equation: If x * 3 = 7, then x = 7/3.

The statement 2 * 3 + 1 = 7 describes a bundling where 2 3-bundles and 1 single are rebundled to 7 1s. The equation x * 3 + 1 = 7 describes the reversed bundling asking how 7 1s can be rebundled in 3s leaving 1 unbundled. Obviously, we first take the single unbundled away, 7 – 1, and then bundle the rest in 3s giving the result x = (7-1)/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 + 1 = 7, then x = (7-1)/3.

A multiple calculation x * 3 + 2 can be reduced to a single calculation by inserting a 'hidden parenthesis': x*3+2 = (x*3)+2. This calculation consists of two steps: First x is multiplied by 3 to x*3. Then 2 is added to x*3+2, which is the total 14. Reversing the calculation consists of the two opposite steps: First 2 is subtracted from the total 14 to 12. Then 12 is divided by 3 to 4.

So a reversed calculation has two sides, a forward side, that cannot be performed because of the unknown quantity x; and a backward side that can be performed by moving the numbers from the forward side to the backward side reversing calculation signs. In the beginning an equation can be solved by walking forward and backward. Later just the numbers walks from one side to the other.

Calculation direction:	Forward		Backward
End	(x*3)+1	=	7
		+1 ↑ ↓ –1	
	x*3	=	7–1 = 6
		*3 ↑ ↓ /3	
Start	x	=	6/3 = 2

The 'Walk&Reverse' method

Forward		Backward
(x*3)+1	=	7
x*3	=	7–1 = 6
x	=	6/3 = 2

The 'Move&Reverse' method

With ten as bundle-size the addition 3 + 5 = 8 predicts the result of adding 3 to 5 by counting-on 6, 7, 8. In the same way multiplication predicts repeated addition of the same number: 3*2 predicts 2+2+2. And power predicts repeated multiplication with the same number: 2^3 predicts 2*2*2.

Any calculation can be turned around and become a reversed calculation predicted by the reversed operations: the answer to the reversed calculation 6 = 3 + ? is predicted by the reversed operation to plus, minus, i.e. by the calculation 6-3. The answer to the reversed calculation 6 = 3 * ? is predicted by the reversed operation to multiplication, division, i.e. by the calculation 6/3. The answer to the reversed calculation 7 = ? ^ 3 is predicted by the reversed operation to exponent, root, i.e. by the calculation $\sqrt[3]{7}$. The answer to the reversed calculation 7 = 3 ^ ? is predicted by the reversed operation to base, log, i.e. by the calculation $\log_3(7)$ = log7/log3.

So the natural way to solve equations is to move numbers over changing their calculation signs:

3 + x = 7 x = 7 - 3	3 * x = 7 x = 7/3	x^3 = 7 x = $\sqrt[3]{7}$	3^x = 7 x = $\log_3(7)$

The Nature of Formulas

The recount-formula and the rest-formula can be used to predict the result of a recounting-process. Recounting 9 in 4s the calculator says 9/4 = 2.25. Here only the left part can be trusted since the decimals are valid only when bundling in tens, and here we bundle in 4s. So the decimal is found from the rest-formula: R = 9 – 2*4 = 1, predicting the recount-result to be 9 = 2 4s + 1 1s = 2.1 4s.

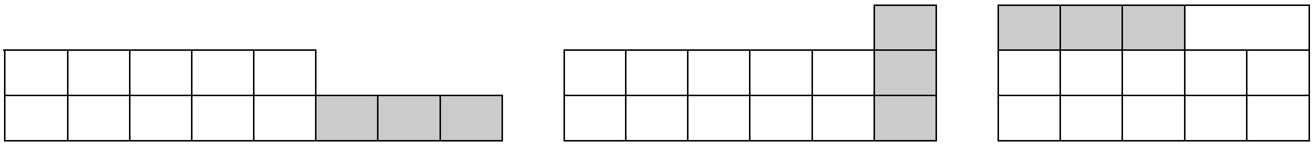
Formulas as number-predictor shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the ‘laboratory’. Thus to avoid collapsing, the dimensions of physical constructions can be predicted by using formulas before being built. And historically religious belief was replaced with predictions with Newton’s three contributions. The Kepler laws about the orbits of planets could not be tested by sending up new planets. Newton saw that the moon is falling towards the earth following its own will, enabling him to test his hypotheses on falling apples. Newton found the formula for this will, and saw that a will causes changes, so therefore he developed formulas for calculating changes.

A formula ‘T = a+b*c’ tells how a quantity T can be predicted by a calculation a b*c. With one unknown, a formula becomes an equation, that can be solved by reversing the calculation, i.e. by moving numbers to the other side changing their calculation signs; and the result can be tested by the ‘math solver’ on a Graphical Display Calculator, a GDC. Containing two unknowns, a formula becomes a function, that can be illustrated as a graph on a GDC; and where the two typical questions ‘given x find y’ and ‘given y find x’ reduces the function to an equation where the solution can be tested by the ‘math solver’, the ‘trace’ and the calc intersection’ on a GDC.

A calculator, however, cannot be trusted when using multiplication. 3*7 means a stack of 3 7s, i.e. 3 7-bundles. 3*7 = 21 only when rebundling in tens, and even here the statement is not completely true since rebundling 3 7s in tens gives T = (3*7)/ten*ten = 2.1 tens, or 21 if we omit the unit ‘tens’ tacitly accepting that when nothing is specified, ten is the unit used for bundling. The knowledge that 3*7 = 21 is not needed for using the calculator to predict the result of rebundling 3 7s in e.g. 5s: T = 3 7s = 3*7 = (3*7)/5 *5 = 4*5 + R where R = 3*7 – 4*5 = 1, so T = 3 7s = 4.1 5s.

The Nature of Fractions

When counting by bundling & stacking, decimals and fractions are parallel ways of treating the unbundled singles. Thus counting in 5-bundles, 3 leftovers can be placed as a stack of 1-bundles next to the stack of 5-bundles and reported by a decimal number as 2.3; or they can be counted as 5s and put on top of the stack of 5-bundles giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.



T = 2*5 + 1*3 = 2.3 *5 = 2 3/5 * 5

Introducing physical units creates ‘double-counting’ counting a quantity in both kg and $: If 4 kg cost 5 $, then the unit-price is 5$ per 4kg, i.e. 5$/4kg = 5/4 $/kg. And the per-equation ‘4kg per 5$’ is used when recounting the actual kg-number in 4s, and recounting the actual $-number in 5s:

10kg = (10/4) * 4kg = (10/4) * 5$ = 12.5$, and 18$ = (18/5) * 5$ =(18/5) * 4kg = 14.4kg.

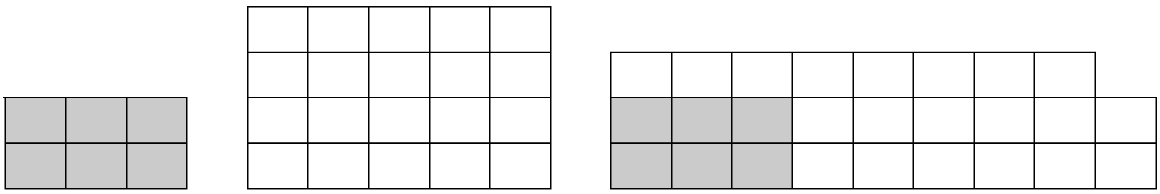
Fractions also occur when splitting a total in parts asking e.g. what is 3 8parts of 200$. Here double-counting produces the per-equation ‘200$ per 8parts’ allowing to recount the parts in 8s:

3 parts = (3/8) * 8parts = (3/8) * 200$ = 75$

The Nature of Adding Fractions is Calculus

When counting by bundling & stacking, the statement 2 * 3 + 4 * 5 = 3.2 * 8 describes a bundling where 2 3-bundles and 4 5-bundles are rebundled in the united bundle-size 8s. This is 1digit integration. The equation 2 * 3 + x * 5 = 3.2 * 8 describes the reversed bundling asking how 3.2 8s can be rebundled to two stacks, 2 3s and some 5s. This is a 1digit differential equation solved by performing 1digit differentiation:

If 2 * 3 + x * 5 = 3.2 * 8, then x = (3.2 * 8 – 2*3)/5 = (T – T1)/5 = ΔT/5



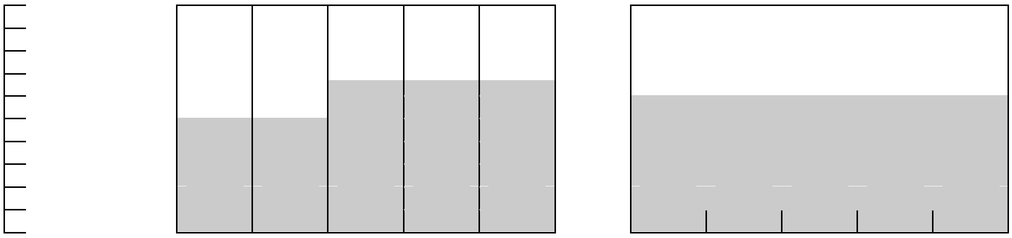
2 3s + 4 5s = 3.2 8s

When adding fractions, it is important to include the units to avoid scaring the learners with mathematism as when performing the following ‘fraction test’ the first day in high school:

The teacher:	The students:
Welcome to high school! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

Seduction by mathematism can be costly as witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. So to add numbers the units must be included, also when adding fractions. And adding fractions f is the root of integration:

$T = 1/2 *2 + 2/3 *3 = \Sigma (f*\Delta x)$, later to become $\int f\, dx$



T = 1/2 *2 + 2/3 *3 = 3/5 *5

The Choices of Modern Pastoral Algebra

Modern algebra still believes in the existence of sets despite of Russell's paradox. It sees its task to present numbers and operations as examples of sets. Thus it defines the set Q of rational numbers as a set of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [Peano-numbers]]]; such that the number (a,b) is equivalent to the number (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number ½. Real numbers are defined in a similar way (see any textbook in modern mathematics, e.g. Griffith et al, 1970).

Having defined the different number sets, modern algebra now defines operations as examples of relations between the product set of a number set and the number set itself. Thus addition in a number set N is an example of a subset of the set of (NxN)xN consisting of elements like (2,3,5) reflecting that 2+3=5. Since addition also contains the element (1,4,5) an equivalence set can be formed in NxN containing the elements (a,b) contained in (a,b,5).

An equation 2+3x = 14 is an open statement saying that the two numbers 2+3x and 14 belong to the same equivalence set thus in need of identical operations to stay in the same equivalence set. Using concepts and theorems from abstract algebra, the neutralizing method can now be introduced.

1+3*x	= 7	An open statement
(1+(3*x)) - 1	= 7 - 1	+1 has the inverse element -1
((3*x)+1) - 1	= 6	+ is commutative
(3*x) + (1-1)	= 6	+ is associative
(3*x) + 0	= 6	0 is the neutral element under +
3*x	= 6	by definition of the neutral element
(3*x) * 1/3	= 6 * 1/3	*3 has the inverse element 1/3
(x*3) * 1/3	= 2	* is commutative
x * (3*1/3)	= 2	* is associative
x*1	= 2	1 is the neutral element under *
x	= 2	by definition of the neutral element
L = {x∈R $\mid$ 1+3*x = 7} = {2}		The solution set

The Arabic Meaning of Algebra

In Arabic, Algebra means ‘reuniting’. With constant and variable unit numbers and per-numbers providing four different kinds of numbers, there are four different ways of uniting numbers.

Addition unites variable unit-numbers, multiplication unites constant unit-numbers, power unites constant per-numbers and integration unites variable per-numbers.

Thus algebra is generated by the four basic questions:

- What is the total of 3$ and 4$’ or ‘T = 3+4
- What is the total of 3$ 4 times’ or ‘T = 3*4
- What is the total of 3% 4 times’ or ‘1+T = (103%)^4
- What is the total of 5 seconds at 3 m/s increasing to 6m/s’ or $\Delta T = \int (3 + (6-3)/5 * x)\, dx$.

Algebra unites & splits	Variable	Constant
Unit-numbers m, s, kg, $, …	T = a + x T – a = x	T = a * x T / a = x
Per-numbers m/s, $/kg, m/100m = %	$T = \int a\, dx$ dT/dx = a	T = a ^ x $\sqrt[x]{T} = a$ $\log_a(T) = x$

Grounded Algebra in Primary School

In primary school the tradition sees the purpose of algebra to be teaching the general rules for adding natural numbers, i.e. the commutative, the associative and the distributive laws.

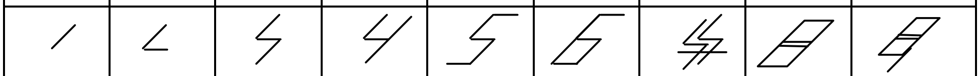
The hidden grounded alternative respects the original Arabic meaning of the word ‘algebra’, reuniting numbers, and the original Greek meaning of the word ‘mathematics’, knowledge that can predict. And it respects a grounded approach to the question ‘what are natural numbers?’

The tradition introduces the natural numbers using the follower-principle, leading directly to the ‘fact’ that 10 is the follower of nine. However, there is nothing natural about 10 as the follower of nine. Counting in 7-bundles, 10 is the follower of six, and the follower of nine is 13.

So a grounded approach to primary school algebra respects that the natural numbers are stacks of bundles as e.g. 3 4s, generated by a counting process predicted by the recount-law T = (T/b)*b iconizing that a total T is counted by talking away bs T/b times.

Thus in primary school, mathematics is learned through counting and recounting by bundling and stacking, and through experiencing that operations predicts a counting result.

1.order counting counts in 1s by rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created until ten.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

To avoid introducing too many number-cons, 1.order counting is soon replaced with 2.order counting, counting by bundling&stacking using icon-bundles: First the sticks are bundled in e.g. 3-bundles, in 3s. Then the bundles are stacked in two stacks: a stack of 3s, and a stack of singles.

3.order counting counts in tens, the only number with its own name but without its own icon.

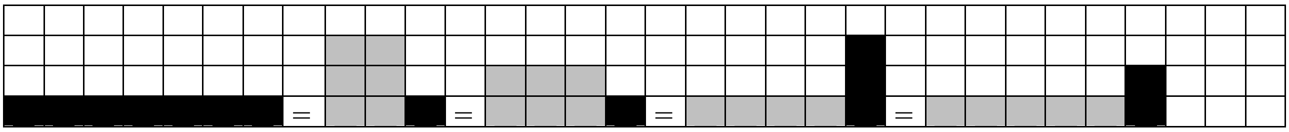
Introducing numbers by the follower-principle, the tradition skips both 1.order and 2.order counting and goes directly to 3.order counting by introducing 10 as the follower of 9; thus the tradition neglects the many leaning opportunities hidden in 1.oder and 2.order counting.

Using 1.oder counting, the learner experiences that the numerals are not just arbitrary symbols but icons written in a more or less sloppy way, in contrast to letters that are pure symbols.

Using 2.oder counting, the learner experiences that a given total can be counted in many different ways. Thus recounting in e.g. 3s, a total of seven sticks gives 2 3-bundles and 1 unbundled single. The stacks may then be placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents can be described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

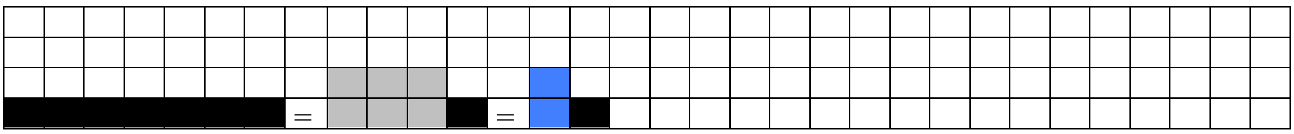
I I I I I I I -> III III I ->	III III I	-> 𝍸 𝍸) I) -> □ □) I) -> II) I)
Or with icons:		-> 2 3s + 1 1s -> 2*3 + 1*1 -> 2)1) -> 2.1 3s

However, a total of 7 1s can also be counted and cup-written and decimal-written in other ways:
T = 7 1s = 3)1) or 3.1 2s = 2)1) or 2.1 3s = 1)3) or 1.3 4s = 1)2) or 1.2 5s = … =)7) or 0.7 8s etc.



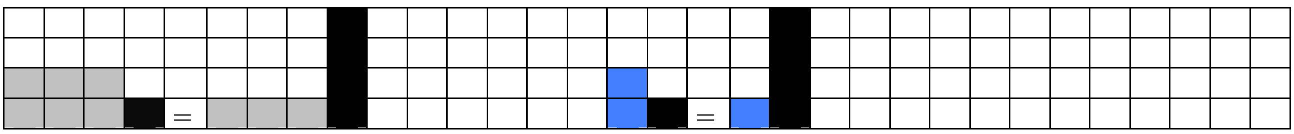
7 1s = 3.1 2s = 2.1 3s = 1.3 4s = 1.2 5s

Using squares, colors can represent the bundles, still maintaining cup-writing an decimal-writing:



7 1s = 2 3s + 1 = 2) 1) = 2.1 3s

Once decimal-numbers become the natural way to describe the result of a counting process, a new activity can take place where ‘overloads’ are created or removed by transforming a bundle into unbundled singles and vice versa. Thus 2.1 3s is the same as 1.4 3s and vice versa.



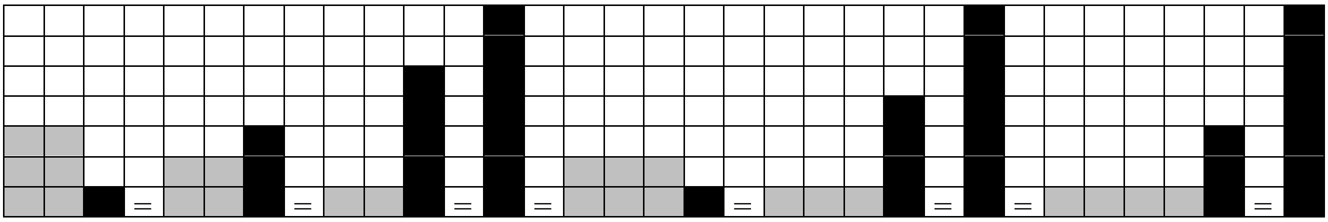
2 3s + 1 = 1 3s + 4 1s 2) 1) = 1) 4) or 2.1 3s = 1.4 3s

Using cup-writing, 1 3-bundle moves from the left bundle-cup to the right single-cup as 3 1s:

2) 1) = 2-1) 1+3) = 1) 4)

Using overloads allows a total of 7 1s to be decimal-written in many different ways:

T = 7 1s = 3.1 2s = 2.3 2s = 1.5 2s = 0.7 2s = 2.1 3s = 1.4 3s = 0.7 3s = 1.3 4s = 0.7 4s = 0.7 8s

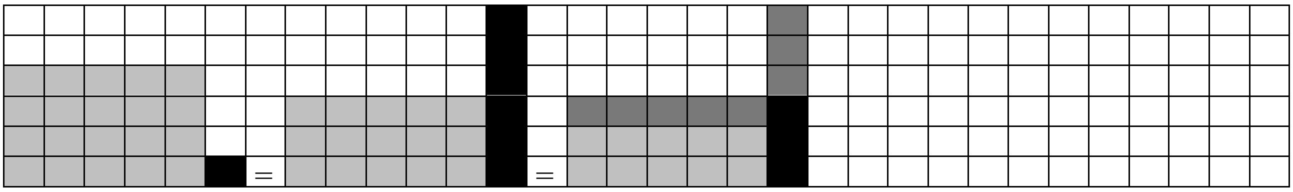


3.1 2s = 2.3 2s = 1.5 2s = 0.7 2s = 2.1 3s = 1.4 3s = 0.7 3s = 1.3 4s

Overloads might occur when selling or subtracting a stack from another stack, e.g. selling 2.3 5s from 4.1 5s, i.e. asking for the result of the subtraction 4.1 5s – 2.3 5s. Here the first stack is rewritten as an overload to 4.1 5s = 3.6 5s. Now

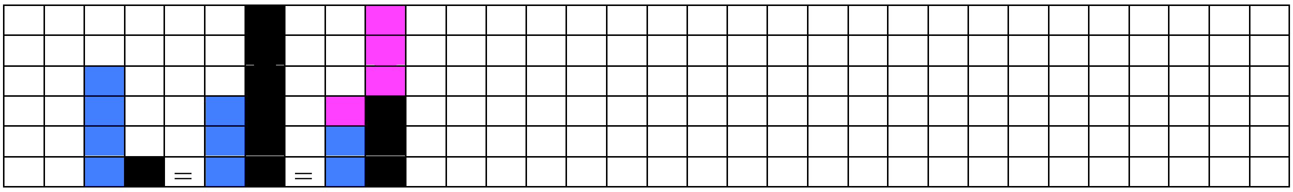
4.1 5s – 2.3 5s = 3.6 5s – 2.3 5s = 1.3 5s, or using cup-writing:

4) 1) – 2) 3) = 3) 6) – 2) 3) = 3-2) 6-3) = 1) 3)



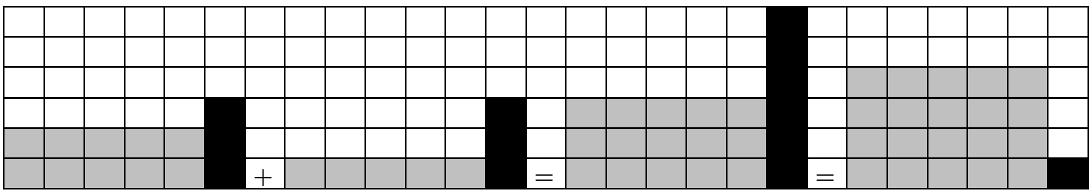
4.1 5s = 3.6 5s = 2.3 5s + 1.3 5s

Or with colors:



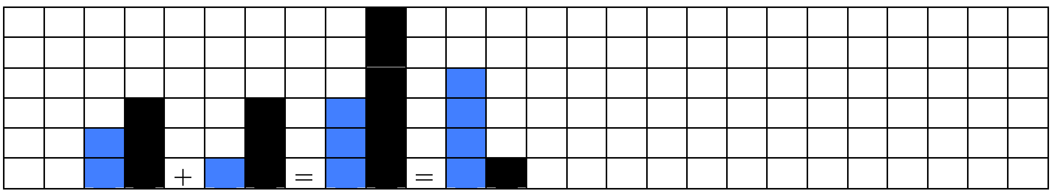
4.1 5s = 3.6 5s = 2.3 5s + 1.3 5s

Also adding one stack to another might lead to an overload that needs to be rebundled. Thus adding the two stacks 1.3 5s and 2.3 5s produces the stack 3.6 5s that can be rebundled to 4.1 5s.



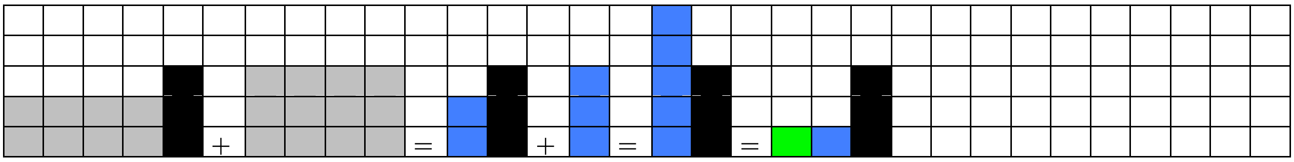
2.3 5s + 1.3 5s = 3.6 5s = 4.1 5s

Or with colors:



2.3 5s + 1.3 5s = 3.6 5s = 4.1 5s

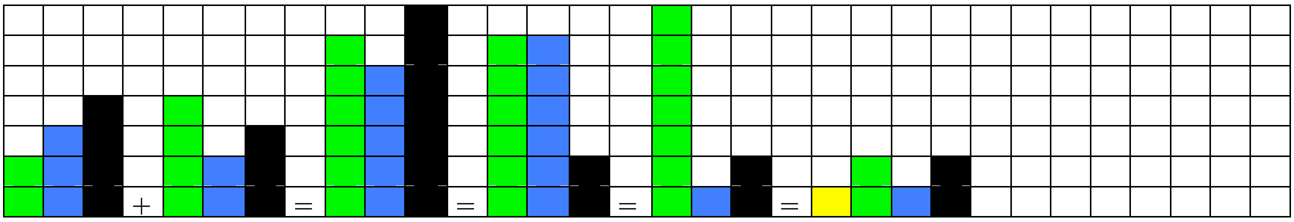
Likewise, 2.3 4s + 3 4s gives 6 4s where the 4 4s can be rebundled into 1 4 4s, thus calling for an extra cup to the left for the bundles of bundles.



2.3 4s + 3 4s = 2) 3) + 3)) = 5) 3) = 1) 1) 3) = 11.3 4s

Since addition implies overloads, also bundles of bundles can be bundled, calling for extra cups.

Thus 23.4 5s + 42.3 5s = 65.7 5s = 121.2 5s:



2) 3) 4) + 4) 2) 3) = 6) 5) 7) = 6) 6) 2) = 7) 1) 2) = 1) 2) 1) 2) = 121.2 5s

Using cup-writing, multiplication can be performed directly, followed by a re-bundling:

2* 34.2 6s = 2* 3)4)2) = 2*3) 2*4) 2*2) = 6)8)4) = 7)2)4) = 1)1)2)4) = 112.4 6s.

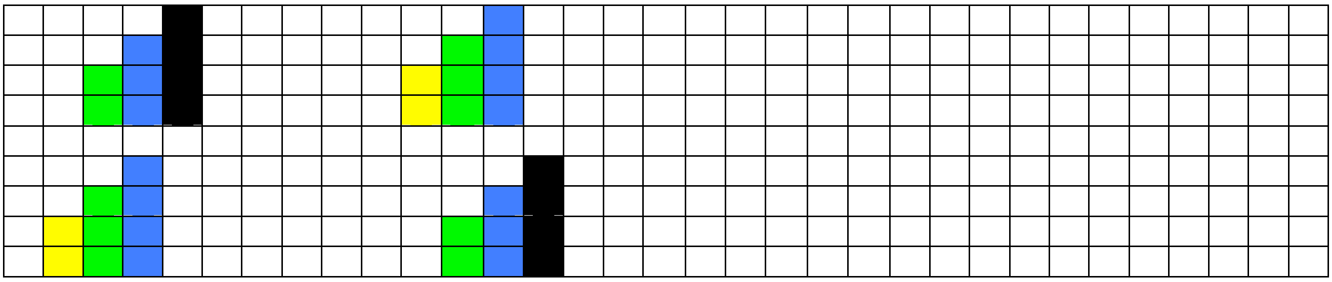
Likewise with division where 34.2 6s / 5 = 4.2 6s + 4 leftovers:

34.2 6s = ? 5S: 3)4)2) = 3*6+4) 2) = 4*5+2) 2) = 4*5) 2*6+2) = 4*5)2*5) + 4) = 5 * 4)2) + 4)

Multiplying a stack with the bundle-number, e.g. 5, just means moving all the stacks to the left, or the decimal to the right. Likewise, dividing a stack with the bundle-number just means moving all the stacks to the right, or the decimal to the left, as seen when using cup-writing:

5 * 2)3)4) = 5*2) 5*3) 5*4) = 2)3)4)0), and

2)3)4)0) = 5*2) 5*3) 5*4) = 5 * 2)3)4), so 2)3)4)0) / 5 = 2)3)4)



23.4 5s * 5 = 234 5s 234 5s /5 = 23.4 5s

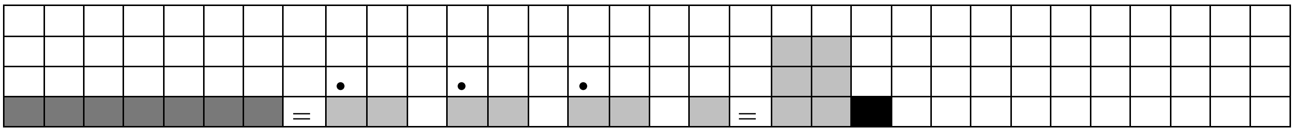
Using cup-writing for addition and subtraction we need not worry about overloads or ‘underloads’, we just rebundle. Thus in the case of 5s the results of the calculations 23.4+42.3 and 123.1–34.2 are

Using 5-bundles:	Using 5-bundles:	Using ten-bundles:	Using ten-bundles:
2)3)4) + 4)2)3) 6)5)7) 6)6)2) 7)1)2) 1)2)1)2)	1) 2) 3) 1) – 3) 4) 2) 1)-1)-1)-1) 0) 4)-1)-1) 3) 4)-1) 3) 3) 4)	7) 8) 9) + 8) 5) 6) 15)13)15) 15)14) 5) 16) 4) 5) 1) 6) 4) 5)	7) 3) 5) – 4) 5) 6) 3) -2) -1) 2) 8) -1) 2) 7) 9)

Cup-writing can also be written down horizontally:

2)3)4) + 4)2)3) = 6)5)7) = 6)5+1)7–5) = 6)6)2) = 6+1)6–5)2) = 7)1)2) = 0+1)7–5)1)2) = 1)2)1)2).

The result of a bundling&stacking process can be predicted by the recount-formula T = (T/b)*b, iconizing that from the total T b-bundles can be taken away T/b times:



T = 7 = (7/2) 2s + 1 = 3*2 + 1 = 3 2s + 1 = 3.1 2s

The rest is found by using the Rest formula R = T – n*b iconizing that the rest is what is left when the stack is removed from the total.

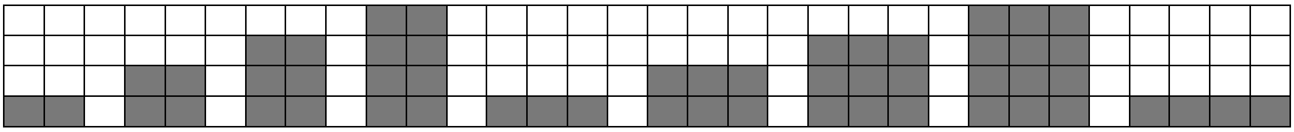
Thus to predict the result of recounting 7 1s in 2s can be predicted by the two formula:

T = 7 = (7/2)*2 = 3*2 + 1 = 3.1 2s since R = 7 – 3*2 = 1. Here the 3 is found on a calculator acting as number-predictor, as the number in front of the decimal point.

Now systematically 1 2s, 2 2s, 3 2s, 4 2s and 5 2s can by recounted in 3s and 4s and 5s etc.

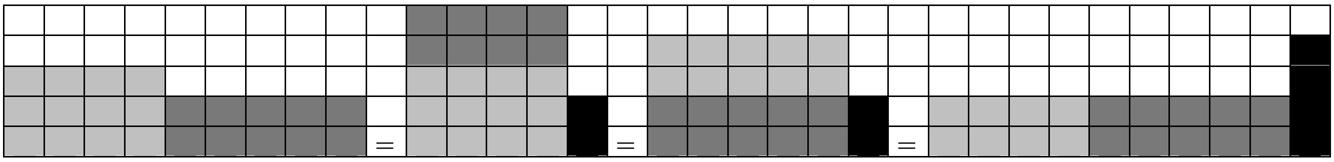
Likewise, 1 3s, 2 3s, 3 3s, 4 3s and 5 3s etc. can by recounted in 4s and 5s and 6s etc.

Thus T = 4 2s = 2.2 3s = 2.0 4s = 1.3 5s etc.



1 2s 2 2s 3 2s 4 2s 1 3s 2 3s 3 3s 4 3s 1 4s

As to adding, the stacks 3 4s and 2 5s can be added in three different ways, as 4s, as 5s or as 9s. In the case of 4s the 2 5s must be recounted in 4s. In the case of 5s the 3 4s must be recounted in 5s. In the case of 9s the two stacks are juxtaposed next to each other, and the surplus is placed as 1s. This might be called stack-integration by juxtaposition since it is a case of ordinary integration.



3 4s + 2 5s = 5.2 4s = 4.2 5s = 2.4 9s

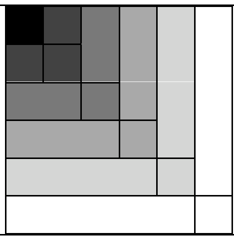
Reversing integration of stacks creates differentiation of stacks. When asking 3 4s + ? 5s = 2.3 9s, first the 3 4s is removed from the 2.3 9s, then the rest is counted in 5s: (2.3*9 – 3*4)/5 * 5 = 1.4 5s = (T – T1)/5 * 5 = ΔT/Δx * 5, which later can be generalized to the rate of change.

T = 2.3 9s = T1 + (T – T1)/5 * 5 = 3 4s + 1.4 5s

When looking at full stacks as 1 1s, 2 2s, 3 3s etc. a pattern occurs when observing that adding two times the bundle-size plus one corner will produce the following full stack:

1 1s

+ 2.1 1s
= 2.0 2s
+ 2.1 2s
= 3.0 3s
+ 2.1 3s
= 4.0 4s



Discovering that n*n + 2.1*n = (n+1)*(n+1)

Grounded Algebra in Middle School

In middle school, the tradition sees the purpose of algebra to be teaching the general rules for adding fractions, i.e. laws for factorizing numbers and polynomials.

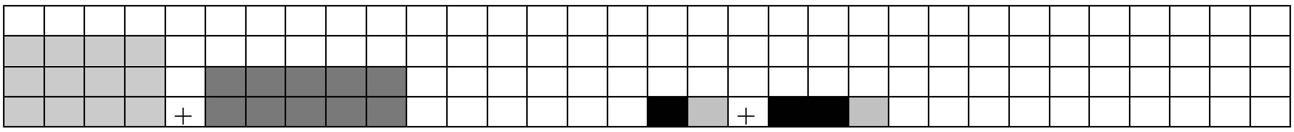
A grounded approach to fractions goes through double-counting, counting a quantity in two different units. A total of 3 can be counted in 1s as T = 3*1, or in 5s as T = (3/5)*5. Thus a grounded alternative does not see fractions as numbers but as operators needing a unit to be meaningful, 2/3 of 3, 2/3 of 7, etc. And a total of apples can be counted in kgs and in $s, resulting e.g. in 4$ per 3kg. The recount-formula from primary school now becomes a per-formula translating kgs to $s and vice versa. Thus to find the price for 12 kg, we just recount the 12kgs in 3s: T = 12 kg = (12/3)*3kg = (12/3)*4$ = 16$. Likewise if we want to find how many kgs 20$ will by, we just recount the 20$ in 4s: T = 20$ = (20/4)*4kg = (20/4)*3kg = 15kg.

Percent means per hundred. Thus we can use the recount-formula to translate per 8 to per 100 and vice versa. To ask '3u per 8 = ? per 100', we just recount the 100 in 8s: T = (100/8)*8 = (100/8)*3u = 37.5u per 100 = 37.5%. Likewise, if we ask '20u per 100 = ? per 12', we just recount the 12 in 100s: T = 12 = (12/100)*100 = (12/100)*20u = 2.4 u per 12.

As operators, fractions carry units when added: 1/2 of 2 + 2/3 of 3 = 3/5 of 5

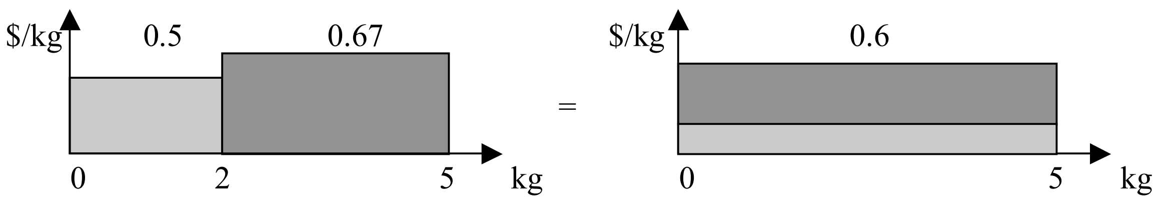
So geometrically, adding fractions is the same as adding stacks by combing their bundles:

3 4s + 2 5s = ? 9s, 3/4 of 4 + 2/5 of 5 = ? of 9.



3 4s + 2 5s = ? 9s 1/2 of 2 + 2/3 of 3 = ? 5s

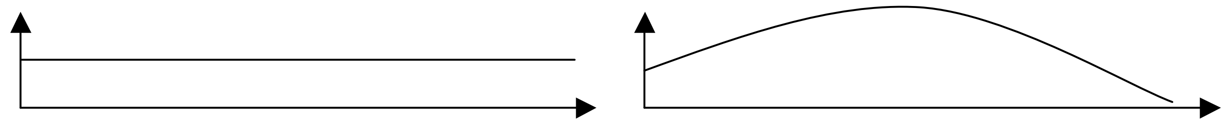
Using decimals instead of fractions, adding fractions becomes a mixture-problem adding quantities with different unit-prices: 1/2 of 2 + 2/3 of 3 becomes 2 kg at 1$/2kg + 3 kg at 2$/3kg, or 2 kg at 0.5 $/kg + 3 kg at 0.67 $/kg = 5 kg at 0.6 $/kg. Here adding fractions means adding the areas under the horizontal fraction-lines.



Grounded Algebra in High School

In high school, the tradition wants algebra to assist calculus in establishing rules for calculating the rate of change Δy/Δx, calling for extensive use of adding and subtracting fractions. The tradition sees calculus as operations on functions, seen as examples of many-to-one relations between sets.

A grounded approach to functions sees the graph as a line that is piecewise or locally constant, enabling the reuse of the mixture-situation where the total is the area under the per-number graph.



Piecewise constant: $\exists\delta>0\ \forall\varepsilon>0: |y-c| < \varepsilon$ in δ Locally constant: $\forall\varepsilon>0\ \exists\delta>0: |y-c| < \varepsilon$ in δ

A grounded alternative respects that algebra means reuniting numbers, in this case per-numbers.

First uniting constant interest rates r into a total rate R leads to a neglected formula 1+R = (1+r)^n, leading to another neglected formula, the saving-formula S/d = R/r predicting the total saving S from a periodical deposit d: If $ d/r is placed on account 1 and the interest r*d/r = d is transferred to account 2, then account 2 contains a saving with a periodical deposit d, but also the total interest of the d/r $ in account 1, so S = d/r*R, or S/d = R/r.

Then variable per-numbers are united when e.g. asking 'what is the total of 5 seconds at 3 m/s increasing to 6m/s' leads to summing up the m/s*s, written as ΔT = ∫ y dx = ∫ (3+ **Error!***x) dx.

It is easy to observe that no matter the number of changes, adding many single changes results in a total change, which can be calculated as the difference between the terminal and the initial values.

y	Δy	ΣΔy
yo = 20		
32	12 = 32-20	12 = 32 – 20
27	-5 = 27-32	7 = 27 – 20
yn = 35	8 = 35-27	15 = yn – yo

In the case of small micro-changes, ΣΔy = yn – yo becomes ∫dy = yn – yo. The change dy can be recounted in x-changes dx as dy = (dy/dx)*dx = y'*dx. So ∫dy = ∫y'dx = yn – yo.

Thus **Error!** = **Error!** = **Error!** – **Error!** = **Error!**, since (**Error!**)' = x^2

The formulas for the rate of change, e.g. dy/dx (x^3), can be found by using 'Calc dy/dx' on a Graphical Display Calculator to set up the results in a table:

x	0	1	2	3	4
dy/dx	0	3	12	27	48

Using regression on a GDC, the formula dy/dx (x^3) = x^2 is generated as a hypothesis that then can deduce predications to be tested on a number of randomly chosen xs:

x	-8	-2	5	28	360	…
dy/dx predicted by the formula (x^3)' = 3*x^2	192	12	75	2352	388800	
dy/dx calculated on a GDC	192	12	75	2352	388800	

In this way we respect the nature of mathematics as a natural science investigating the natural fact many by using the method of the natural sciences to produce knowledge: observe, induce a hypothesis, and deduce predictions for testing the hypothesis.

Conclusion

To deconstruct pastoral choices presented as nature in contemporary algebra, anti-pastoral sophist research has uncovered several hidden alternatives. Historically, the roots of algebra is what its Arabic name says, the task of reuniting numbers. However two kinds of numbers exist, pastoral numbers claiming to be natural; and grounded numbers generated through counting by bundling & stacking as stacks reported by decimal-writing including the unit. Allowing 1.order and 2.order counting to take place before going on to the traditional choice, 3.order counting, means allowing the learners to profit from the many educational activities of 2.order counting, especially re-counting in different bundle-sizes and double-counting in different units. Fractions without units can be deconstructed to show the nature of fractions as operators always carrying units, and that when added with units becomes the root of calculus, which then can be deconstructed to adding variable per-numbers. Also equations seen as equivalence statements about number-names can be deconstructed to show the roots of equations, reversed calculations, which allows the natural method of solving equations, move over and reverse calculation sign, to be introduced as an alternative to the traditional neutralizing method. Finally, to improve the learning of algebra, an alternative algebra curriculum for primary school, middle school and high school is outlined.

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