INEQUATIONS RESOLUTION: A FUNCTIONAL GRAPHIC APPROACH

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In this paper, we present some of the results related to a research study involving 7 Mathematics teachers working with a set of activities on inequations resolution. Activities were inspired by Representation Semiotic Registers Theory (Duval, 1995, 2000) and designed using algebraic, graphic and natural language registers, in a functional graphic approach. Teachers' writing protocols were analyzed under the light of Fischbein's ideas (1993), according to which we must look for formal, intuitive and algorithmic aspects in all mathematical activity. Evidences show that these teachers do not dominate formal aspects at all, including the logical ones. Also, they always use intuitive aspects, mainly numerical ones, almost coercively. So, they couldn't relate both resolutions, algebraic and graphic, in order to discuss problems connected to inequations algebraic resolution.

INTRODUCTION

Inspired by a question posed by Sackur (2004) about the use of functions' graphs to "solve an inequality"

"… what do they learn in Mathematics when they solve algebraically or graphically?" (SACKUR, 2004, p. 1-151).

And by a suggestion made by Dreyfus & Hoch (2004)

"We suggest that the forum address the issue of ways of presenting algebra that will focus students' attention on structure." (DREYFUS & HOCH, 2004, p. 1-155).

We decided to investigate if we could promote learning of *algebraic resolution of inequations* (in this paper, we give the name 'inequation' to 'a inequality with one real unknown' for short) by using registers associated to three different semiotic systems of representation (DUVAL, 1995, 2000), algebraic, graphic and natural language in an approach that we have called *functional graphic generic*.

Functional, because we look at each side of an inequation as a function whose variable is the unknown. So, we have two functions to work with.

Graphic, for the reason that we use the graphs associated to each one of these functions. So, we have two graphs to look at in the way to solve an inequation.

Generic, because we bring assorted functions and graphs into discussion: first-degree and second-degree polynomial functions, function g defined by $g(x) = |x|$ and rational functions as $h(x) = \frac{mx + n}{px + q}$, with m, n, p, $q \in IR$ and $p^2 + q^2 \neq 0$. What's more, in a functional graphic approach we don't need to restrict the function we work with, if we can provide the graph associated to it. What we do need is a clear understanding of function, graph and the language involved in a sentence like r(x)<s(x).

To realise the proposed investigation, we have designed a set of five activities to be applied to and discussed with seven Brazilian Basic School Mathematics teachers, who have worked under the guidance of a group of Mathematics Education researchers (and we are among them) for at least seven years, in weekly meetings of three and a half hours, to discuss Brazilian Secondary and High School Mathematics (11-16 years-old) contents as well as approaches to these ones. We have used 12 of these meetings to develop and discuss all five activities.

To validate possible results we could have, we chose to analise teachers' writing protocols looking for formal, intuitive and algorithmic aspects as well as interactions between them, under the light of Fischbein's main claim (1993) "in analysing the students' mathematical behavior, one has to take into account three basic aspects: the formal, the algorithmic, and the intuitive" (FISCHBEIN, 1993, p. 244). We also draw from this claim the necessity for Mathematics teachers to promote and to valorise, in their classrooms, all three aspects and their interaction.

Although we were dealing with seven Mathematics teachers, with at least seven years of practice in Brazilian Secondary and High School classrooms each one, we could observe, reading their protocols: a complete non preoccupation with formal aspects and many doubts concerning what we have called logical formal aspects; a great emphasis in intuitive aspects, mainly numerical ones; a reasonable understanding of algorithmic aspects in liaison with graphic resolution for the proposed inequations; and a total failure in doing the connections between graphic and algebraic resolutions.

Taking it all, we can say that the biggest trouble is not on the use of various systems of semiotic representation. We see a major difficulty in the way Mathematics is being presented to students: a lot of intuitive and algorithmic aspects and none of formal ones.

In particular, the answers to four questions have caused us to read, in Mariani's thesis (2006), her findings about the influence of Brousseau's Didactic Contract as well as his definitions (BROUSSEAU, 2003).

In this paper, we present the outcomes of these four questions, which we consider significative to exemplify what we have saying so far.

QUESTION

We have put under light if an approach functional graphic generic to inequations' resolution, using three semiotic representation systems, algebraic, graphic and natural language, can help individuals to interact formal, intuitive and algorithmic aspects in liaison with algebraic resolution.

THEORETICAL FRAMEWORK

We have supported our research in two cognitive theories. One in order to design a set of five activities we have applied and observed and the other, to analyse writing protocols we have obtained with these activities' application. The first one is due to Duval (1995, 2000) and is named Representation Semiotic Registers Theory. The second one is due to Fischbein (1994) and it is concerned with aspects we must observe when analysing someone's mathematical behaviour.

According to Representation Semiotic Registers Theory, in thinking, writing or communicating Mathematics, a person must uses various Representation Semiotic Systems, in order to discriminate an object from its representation. A Representation Semiotic System, in this context, means: *one or more signs* that can be associated to anything belonging to the chosen system; a set of rules that allow *compound* these signs in order to obtain a representation and *transform* it to enlighten new knowledge; a set of rules that permit to *convert* representations from a system to another as a means to bring new characteristics about the represented object.

As Duval (1995, 2000), we believe that a subject must use at least two Representation Semiotic Systems in order to succeed in Mathematics, and actually, we have used three of them, **algebraic**, **graphic** and **natural language**.

Algebraic, because is the expected one, and we believe the most efficient.

Graphic, exactly because it is not usual in Brazil and, we think, graphs can help students to compare solutions and see *inequation*'s structure. Also, we think it is possible to surpass graphs' learning difficulties and learn to use … by using them.

Natural language, for the simple reason that, for us, it is a good way to help student to understand inequation's structure, if we permit them to express themselves using natural language.

According to Fischbein (1994), we must analyse students' mathematical behaviour taking into account three basic aspects: **formal**, **intuitive** and **algorithmic**.

Formal aspects are axioms, definitions, theorems and proofs.

Algorithmic aspects refer to solving techniques and standard strategies.

Intuitive aspects are connected to an individual degree of direct acceptance of a notion, a theorem or a solution.

As Fischbein (1993), we believe that an individual knows a subject in Mathematics if he is capable of interacting formal, intuitive and algorithmic aspects that are connected to this subject.

RESEARCH METHOD

We have developed a qualitative research, inspired by our reading of Artigue (1995) about Didactic Engineering, in three main steps: *preliminary analysis*; *conceiving, designing, applying and observing a sequence of activities*; and *validating*.

Preliminary analysis includes: a literature revision; a more profound analysis of some aspects of Representation Semiotic Registers Theory; a reflection about both approaches, algebraic and functional graphic; and an analysis of some Mathematics textbooks for Brazilian Secondary and High School (11-17 years old).

Conceiving, designing, applying and observing a sequence of activities mean the whole work we have done to produce, to apply and to observe our set of activities. Also includes a didactical analysis of each activity.

Validating is essentially our analysis of the written protocols, looking for formal, intuitive and algorithmic aspects we could find in order to answer our question: "Can a functional graphic generic approach to inequations resolution provide the means to algebraic resolution understanding?"

We have made such a choice because we saw our research as an intervention in the researched group and not just as a written report to divulge some quantitative or qualitative results. We hope our set of activities, as well as our preliminary analysis and our validation can be of assistance to these or to other individuals, if they want to use the results we have obtained with our designed, applied and validated sequence about inequations resolution.

SAMPLE DATA

We chose to present in this paper results connected to four significant questions we have made in the 5th activity (the last one). In what follows, we put these four questions texts and each one is illustrated by some answers we selected from teachers' protocols in order to exemplify conclusions we have made so far.

We had all 7 teachers present in this session.

Question 1:

> **"In order to solve inequation $x^2 \leq 9$, a student gave the following algebraic resolution: 'Extracting square root from both sides, we have inequation $x \leq \pm 3$; so, answer to given inequation is $x \leq \pm 3$'.**
> **Observe that, by student resolution, inequation $x \leq \pm 3$ might be equivalent to original one.**
> **Analyse the text and see if you agree with this student. Discuss your doubts."**

We had in mind teachers would reflect about student's resolution and notice that $x \leq \pm 3$ doesn't have proper mathematical denotation.

Only 3 teachers answered this question: 2 of them accepted $x \leq \pm 3$ as a proper denotation, one as $x \leq -3$; the 3rd one writes

> "$x^2 \leq 9$ equivalente? $x \leq \pm 3$."

But doesn't explain if his doubt is in the "equivalent" or in the whole question.

None has formally analised student's resolution or put in words his doubts.

Question 2:

> **"Looking at the graph, determine inequation $x^2 \leq 9$ solution."**

To answer this question, individuals had access to a graph of the function defined by $f(x)=x^2$ on a computer screen, using software Cabri-géomètre II and could use its dynamics to observe graph's points coordinates. It is important to say that these teachers were used to this software and to computational environment.

All 7 teachers looked at the graph to give the answer, but just 6 give it right: 2 of them have it complete; other 4 have it incomplete as in

> "for $-3 \leq x \leq 3$ we have $x^2 \leq 9$."

Or in

> "$S=\{(x,y) \in IR \mid -3 \leq x \leq 3\}$"

The first one has done just an implication and not equivalence. The second one has chosen the set of points as an answer to an inequation.

And teacher with the wrong answer gives $x \le 3$ as the set of solutions.

None of them has tried to compare his answer with the previous one.

Question 3:

"Does your solution coincide with students' solution? Justify your answer."

All 7 teachers use function's graph to say "no", but none formally justifies his answer. They only compare the set of solutions with $x \le \pm 3$: 5 of them accept this as an "or"; one as an "and"; and one as $-3 \le x \le +3$.

Actually, one teacher writes

> "Yes! Algebraic procedure gives the same solution, but it is important to observe … 'ops'! $x \le 3$ $x \le -3$? Reconsider: the reading of $x \le \pm 3$ makes me think that $x \le 3$ or $x \le -3$, therefore it is not equivalent to graphic one."

And don't explain why they are different.

Question 4:

"What is your pupil error? Justify your answer."

1 teacher didn't answer and each one of the others 6 gave a different reason for pupil's error. We bring here five of them.

(1) "$x^2 \le 9$ $x \le \sqrt{9}$ $x \le \sqrt{3^2}$ $x \le \sqrt{9}$ $x \le 3$. The mistake is in saying that square root of 9 is also -3! …"

(2) "Pupil's error is in $x \le \pm 3$."

(3) "Student has extracted square root from both sides, from this on, equations equivalence is lost. $x^2=9 \not\Leftrightarrow \sqrt{x^2}=\sqrt{9}$."

(4) "In using a procedure that works for equations resolution, not always it satisfies all necessary conditions for an inequation. One needs to carefully analyse all implications in certain situations."

(5) "$x^2 \le 9$ $x \le \pm 3$ (+3 ou -3?) ALGEBRAICALLY! If pupil answer was just $x \le 3$, it would be 'more correct'."

None of them has learned that $\sqrt{x^2} = |x|$, although we have worked this with them a little before this question. They can say $\sqrt{9} = 3$ (a numeric fact), but they can't accept $\sqrt{(-3)^2} = +3$. With this, the valid procedure "extract square root from both sides" is misunderstood, since they don't know how to apply the necessary hypotheses as "if *p* then *q*".

RESULTS

Our analysis shows that the individuals of our research don't know how to *compound* the *algebraic system signs* and they act without logical formal aspects, which means that they have difficulty with mathematical phrases as “if p then q”, “p if and only if q”, “p or q”, “p and q”. As a consequence of these difficulties, they couldn’t *read* inequation’s structure, as we have wished from the beginning.

Also, they act almost always on the basis of intuitive aspects that we have called numerical (a kind of urgent necessity in substituting some numerical value or in determining the solution set) and they have a total lack of formal aspects (none has done explanation to the posed questions, even if we have asked for it).

Looking for an explanation for these difficulties in this group of Mathematics teachers, mainly for the lack of formal aspects, we read again answer (5) above and we think we find an answer in classroom’s relationship between teacher, student and Mathematics, under the laws of Brousseau’s Didactic Contract (2003), whose definition is

> “It is the set of reciprocal duties and ‘sanctions’ that each partner in a *didactical situation*
>
> - imposes or believes to impose, explicitly or implicitly, on the others,
> - and those that it is imposed to him or he believes so,
>
> about the knowledge into play. Didactical contract is the result of a ‘negotiation’, frequently implicit, of rules establishing relationships between a student or a students group, a certain environment and an educational system. One can consider teachers duties as a noteworthy part of a ‘didactical contract’, vis-à-vis the society that gives him his didactical legitimacy.
>
> Didactical contract is not a real one for the reason that it is neither explicit nor freely established, and because neither rupture conditions nor sanctions can be done in advance as its didactical nature, that really matters, depends on knowledge still unknown for the students.
>
> Also, it is frequently unsustainable. It puts the teacher in front of a real paradoxical injunction: everything he does in order to obtain what he wants from students, normally diminishes their incertitude and don’t give them the necessary conditions for understanding and learning the desired subject: if the teacher says or explains what he wants, he just has an executed order and not an exercise of their comprehension and their judgement (first didactical paradox). […] But student is also in front of a paradoxical injunction: if he accepts that, under the contract, teacher must give him solutions and answers, he doesn’t establish them by himself and then doesn’t engage them into his mathematical knowledge and can’t take it for his own use; if he wants to learn, he must refuse didactical contract and take in charge, autonomously, the problem. Learning is going to lie on contract *ruptures and its adjustments* and no more on the contract in good working itself. When a rupture occurs (student’s or teacher’s choice), partners’ behavior is as there was a real contract between them. […].” (BROUSSEAU, 2003, our version.)

CONCLUSION

Although these individuals have accepted graphical resolution to the proposed inequations, they had failures in their general mathematical formal and intuitive aspects background in such a way that they were not capable to compare both resolutions, algebraic and graphic, in order to open the discussion about their mistakes and their doubts in algebraic resolution.

The reading of Brousseau’s Contract Didactic definition motivated us to put some questions for further and urgent research.

“Which Contract Didactic is valid in Brazilian Basic School?”

“Which Contract Didactic’s clauses we need in Brazilian Basic School?”

“Why so big the fear in teaching Mathematics formal aspects?”

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