Abstract:
In the Netherlands a discussion is going on about the role of arithmatical and algebraic skills in the ongoing learning lin form primary education to acadamic education and to vocational education.
In this line especially the transition from arithmetics to algebra gives interesting possibilities for mantaining and expanding the arithmecal skill in the learning of algebra. The ReAL-project investigated these posibilties and constructed three (of many possible) learning lines from arithmetics to algebra. These learning lines are described in overview in this paper.
The learning lines are based on gaps that were found in the most commonly used schoolbooks for mathematics. The gaps were translated into changes for improving the learning of algebra and by doing that giving a constructive role to the maintenance of arithmetic skills in secondary education. This translation resulted in the learning lines mentioned above.

Learning Lines: From arithmetic to algebra, from elementary towards secondary education

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Introduction

It is interesting to see how, both nationally and internationally, certain ‘buzz words’ keep coming up in mathematics education. Everybody talks about these issues, not everybody seems to agree on what the buzz words mean but almost everyone expresses strong opinions about them. Students nowadays seem to be lacking in computational skills. However, which students are we talking about and which skills can be expected from them at a certain age and what does it mean to be ‘proficient’? Inadequate algebra skills also keep coming up but should we expect the same algebra skills from a vocational education student, who will start working as soon as (s)he is allowed to and from somebody at a pre university course? “Choice” is a key word in the Dutch educational system and unless we want to change that, we should take into account that different choices will lead to different expectations and demands.
All this being said, it is obvious that for large groups of students problems have been pointed out and these problems should not be neglected. Just saying: “They can’t even........anymore!” won’t help us any further.

Another concept that keeps buzzing around is *learning lines*. In an ideal situation, the (long) learning lines should start in elementary education and continue through middle school and secondary education. But also from the first years of secondary education towards later years. There are many different mathematics learning lines, some of them more detailed than others. In this article, we will limit ourselves to learning lines from arithmetic to algebra and to students from age 12 – 16. In an earlier article, we wrote about the importance of the *fractions learning line*, that is why less attention to this important line will be spent here.

The Dutch ReAL project

In the Dutch ReAL project, two institutions collaborated, SLO (National Institute for Curriculum Development) and Freudenthal Institute for science and mathematics education, university of Utrecht. The (translated) acronym means **A**rithmetic and **A**lgebra **L**earning **L**ine. The part of the project addressed in this article deals with students aged 12 – 16. The project was funded by the Dutch Ministry of Education. Important goals were:

- A description of important concepts needed to acquire skills in arithmetic and algebra as needed to successfully prepare for high school.
- A description of the conceptual networks that students need in order to acquire those skills
- Sample teaching materials together with didactical notes to show how students may learn, maintain, and use these skills in complex situations as well as obtain a deeper algebraic insight needed for further study and occupational use

The Dutch ministry also wanted the project members to indicate levels of understanding at different crucial moments during the learning line from elementary towards secondary education but this proved to be too ambitious for a small project like ReAL. Instead, sample tests were designed, related to important parts of the learning line and allocated to different age groups or grade levels.
Since little time was available for the project, expertise from different sources was used as much as possible. Apart from the project team and their colleagues, an advisory group consisting of teachers, teacher educators, and members of the Dutch association of maths teachers was established. Some of the materials were tried out at pilot schools.

First and foremost, an overview of important focal points was made. Based upon these focal points, (parts) of important learning lines were indicated. Two often used mathematics textbook series were analyzed to establish possible gaps and decide where additional materials might be appropriate. Lessons were observed and with teachers the use of additional materials was discussed. A few schools agreed to use the materials designed within the project and send written reports about their findings. All materials designed within this project were made available through the website www.slo.nl/real (in Dutch).

At the end of school year 2006/2007, the first part of the ReAL project ended. The second part, ending by the end of 2007, focused on the vocational learning trajectory. For this part, arithmetic is more important since in vocational education arithmetic and its use within professional situations gets more attention.

Conceptual network

Just as it is impossible to learn a language by learning single words and grammatical rules only, it is equally impossible to become mathematically literate by learning single definitions and standard algorithms only. A conceptual network is needed in order to establish underlying relationships and coherence. Only then real learning lines are brought to the fore for the students. The tests made within this project try to reveal whether or not students have such a conceptual network at their disposal.

Next we will show an example from the sub domain of *fractions*, from the test designed for elementary students (aged 12 years) that will later follow a pre university track or students at their first year of this track (aged 12 years).

1. What is more, $\frac{1}{7}$ or $\frac{1}{8}$? How do you know?
2. The letter n represents a whole, positive number. What is more, $\frac{1}{n}$ or $\frac{1}{n+1}$? How do you know?

The results from a small pilot group showed that most students were perfectly able to show that $\frac{1}{7}$ is greater than $\frac{1}{8}$. They obviously learned a standard procedure and compared both fractions by using equal denominators: $\frac{1}{7}=\frac{8}{56}$ and $\frac{1}{8}=\frac{7}{56}$.
If apart from the standard algorithm, the appropriate *conceptual network* is available to these students, by reasoning that $\frac{1}{7}$ is more than $\frac{1}{8}$ since the first fraction has the larger denominator ("you get a larger piece of the pie"), they can use this concept to deduce a general rule and state that $\frac{1}{n}$ is greater than $\frac{1}{n+1}$ for the same reason. If this conceptual network is lacking, the students have no ways to conclude that in the general case $\frac{1}{n}$ is greater than $\frac{1}{n+1}$ since the denominator in the second fraction became one more. At this age they do not have the algebraic knowledge needed to find equal denominators for both fractions and compare them. In this case this is not a desired strategy either, since it is needlessly laborious.

Analysis of frequently used text books

Below a few of the remarkable points regarding learning lines from arithmetic to algebra are summarized as a result of our analysis of frequently used mathematics text books and comparable sources.

- Arithmetic, and more specific developing knowledge about number systems and computational rules, does not get much attention in pre university courses (starting at age 12)
- Algebra is not presented as a continuation, generalization, or abstraction of arithmetic or in any case hardly so. No deliberate connection to what was taught in elementary education is visible.
- Elementary algebraic competences are connected to the shape of algebraic equations and forms. Looking at the meaning of numbers and variables hardly ever occurs, at least not structurally and supporting mathematical models are seldom used
- No or hardly any attention is paid to constructing formulas and manipulating with formulas or other algebraic forms
- Many opportunities for additional practice and rehearsing algebraic procedures are offered

Teachers certainly do recognize the importance of developing en maintaining computational and algebraic skills based on understanding. However, it is difficult to help developing a critical attitude and teach students to reason mathematically and discuss strategies in class if their students, as is often the case in The Netherlands, mainly work independently following their text book. For this reason, after discussing these issues with participating schools, some additional materials were developed to fill the gaps that were indicated by the teachers. During this small project, only very limited experiences could be established and because of that no firm conclusions can be drawn from these experiences. Only during a much longer project can be established whether or not students develop mathematical skills as well as mathematical understanding when using algebraic expressions if there is a focus on the structure of expressions and formulas, if supporting mathematical models are used and if making a formula yourself, based on the information provided, is stressed from an early stage onwards. In other words, if not only using algebra rules but more specific algebraic thinking is developed.
Other observations show that students who just get more practice using standard rules that are misunderstood or not understood at all, do not favor from this extra practice and do not get better skills.

What did we notice in the mathematics text books used in elementary education?

The text books used in elementary education (students age 5 – 12 years old) have been studied less systematically and less intensively because this went beyond the assignment of the ReAL project. However, a few remarkable observations may be noted here that teachers in secondary often are not aware of and for that reason do not take into account.

- In elementary education, students do not encounter larger context problems where students first answer an introductory question (a) and next some follow up questions (b, c, d, ...) that require an increasing level of problem solving skills. This results in a way of dealing with each separate question as if it is ‘on it’s own’ and has nothing to do with the context problem as a whole.
- Hardly ever students are being asked: *Show your reasoning* or *Show how you found your answer*. In text books for elementary education only the *product* seems to be of importance and not the *process*. In secondary education it is the other way around.
- Rules are looked upon as a *recipe to find the right answer* and not as *generalizing informal strategies*. Teachers should more often ask: “Is that always so?” “How can you be sure?”

In elementary education the focus is on *learning to do computations* and less emphasis is placed on using arithmetic in context situations or applying the rules that were learned for problem solving.

All of this does not mean elementary teachers do a bad job. What we see in the text books are the problems and questions meant for the students to practice what the teacher taught. The explanations from the teacher, the classroom discussions, and the work in separate groups are not shown by the text book. Even than, in our view, an informal introduction to algebra which is now neglected, should take place in elementary education already. It is not much use to show that negative numbers exist by showing one example in the book (negative numbers on a Celsius thermometer) and afterwards only once in the book using negative numbers in a coordinate system.

If Celsius thermometers are used in your country and some day it is freezing, this is a good time to discuss the existence of numbers below zero. But negative numbers may also show up in an natural way during a classroom discussion when 'counting backwards from 100 in jumps of seven' is practiced. Does a student stop at two or does (s)he continue?
Another student may write 28 – 74 = ..., instead of 74 – 28 = ..., which is a common mistake. This may lead to a discussion about numbers below zero again. Why can't we do the first problem? Do negative fractions exist? Could decimal numbers ge below zero on the number line? Etcetera.
Certainly there must be teachers using these possibilities, but that does not create a learning line. For that systematical attention is necessary in or in addition to the schoolbooks.

Projectproducts

Learning lines are written on where we saw gap in the schoolbooks.

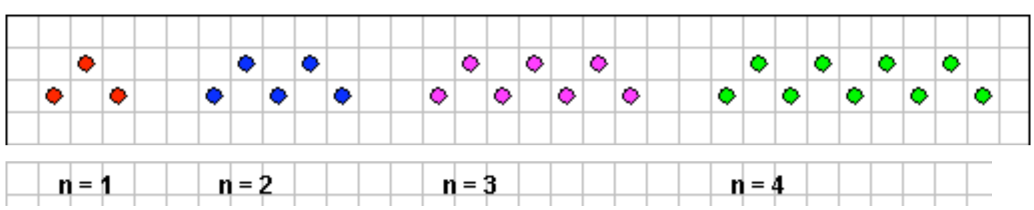
The *fractionline*. In our vision the operations with fractions should transfer in analog actions with fractions with variables. And the transfer of written language to formal algebraic language should het attantion.
Halve of five can be written as $\frac{1}{2} \times 5$; as 5 : 2, and as $\frac{5}{2}$.
Halve of $\frac{2}{3}$ is $\frac{1}{3}$, so halve of $\frac{2}{n}$ is $\frac{1}{n}$.
The *multiplicationline* shows how multiplication of number can be connected to multiplication of algebraic expressions without loss of insight. This can be illustrated with help of a thinking model; the so called rectangle- or areamodel with transfer to the use of the more abstract table later on in the learningproces. This model is also choosen because it can give a clear illustration of the distributive en associative property.

In the world around us one can find many patterns and relations with an arithmetical relation between variables. Finding these arithmetical relations and formulate them with help of the language of algebra is the essence of the *formulaline*.

We give one example of a problem in the formulaline. Problems of this type, with special attention for examining structures are rare in the Ducth school books for mathematics.

Dotspattern

4.



a. Show (e.g. in a drawing), that each of these formulas fits to this dotspattern. I each formula stands s for the number of dots and n for the number of the pattern.

$$s = (n + 1) + n$$
$$s = 3 + (n - 1) \times 2$$

$s = 3n - (n - 1)$

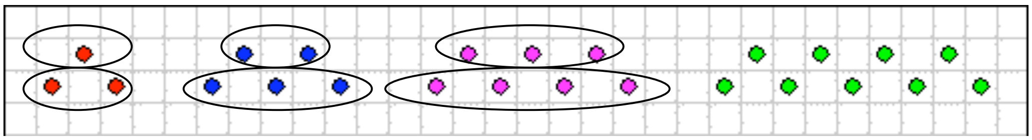
b. Write all three formulas as short as possible.

c. Show how one can see the short formula in the dotspattern.

Possible answers:

$s = (n + 1) + n$

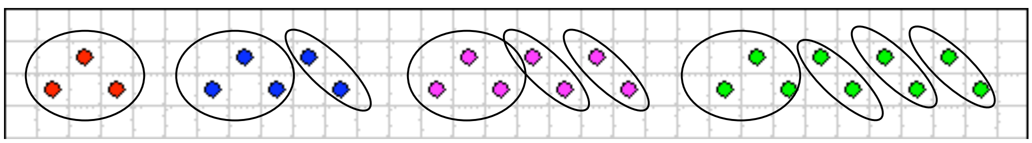
Drawing:



Explanation: lower row $n + 1$ dots, upper row n dots

$s = 3 + (n - 1) \times 2$

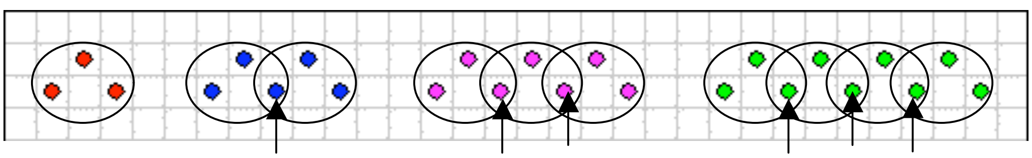
Drawing:



Explanation: Three dots in a triangle plus 'one-less-than-n' sloping rows of 2 dots.

$s = 3n - (n - 1)$

Drawing:

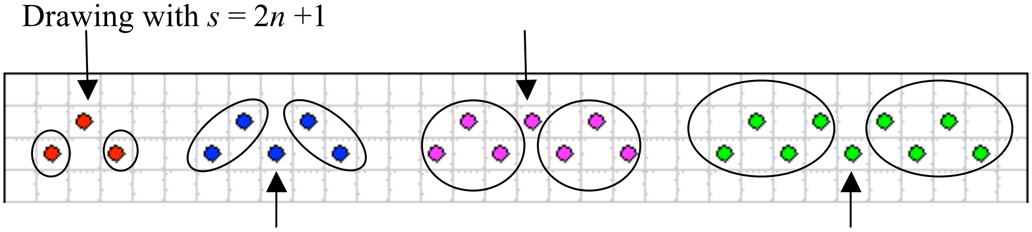


Explanation: n times a triangle (so n times 3 dots), minus the double counted dots (those are n-1).

b. Al formulas can be simplified to $s = 2n + 1$

Several algebraic skills are involved: adding up terms, replacing terms (commutative property), usib bracets, using correctly minus-signs en bracets.

c. Drawing with $s = 2n + 1$



Conclusions

In the commonly used schoolbooks for mathematics we find the following:

- They offer many opportunities to train and maintain algebraic skills, however arithmetical skill don'get much attention.
- In general, algebra is not or only marginally presented as a continuation, generalisation or abstraction of artithmetics.
- Elementary algebraic skills are connected to outer characteristics of algebraic expressions. *One can add up two terms once the parts with the letters are the same.* To give meaning to these action by using for instance models is seldom and certainly not structurally done.
- There is little to none attention for construction the students own formulas and manipulating them.

So on both sides of the line between primary and secondary education a lot can be learned about the connecting learning lines, especially in the area of arithmetics and algebra.

References:

Burkhardt, H.(2001). *Algebra for all: what does it mean? How are we doing?* In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *The future of teaching and learning of algebra* (Proceedings of the 12th ICMI study Conference, pp. 140-146). Melbourne, Australia: The university of Melbourne.