Algebraic procedures used by 14 to 15 year old Sri Lankan students

Gunawardena Egodawatte
Department of Curriculum, Teaching and Learning, University of Toronto, Toronto, ON, Canada.
Phone: 416 413 0280 E-mail: egunawardena@oise.utoronto.ca

Abstract

Research studies have shown that students encounter difficulties in transitioning from arithmetic to algebra. Especially, understanding the structural aspect of algebra is more difficult for students than the procedural aspect. A sample of Sri Lankan high school students was given an algebra test followed by individual interviews to identify error patterns and their causes in algebra. An interview method developed by Newman and others was used for this purpose. This article explains the types of procedures that were used by the students and how they explained their reasoning during the interviews. Six prominent error types were identified among others and three of them are discussed here with possible causes. The origins of errors were: intuitive assumptions and failure to understand the syntax of algebra, analogies with other familiar symbol systems like the English alphabet, interference from other learning in mathematics such as arithmetic, and other psychological factors such as carelessness or lack of motivation.

Introduction and background

Some algebra is taught to all junior secondary students starting at about 11 years of age, depending on the country and program. MacGregor (2004) listed the reasons as to why students should learn algebra. First, algebra is a necessary part of the general knowledge of members in a democratic society. Second, it is a prerequisite for further study of mathematics. Third, it is a crucial component of mathematical literacy. Fourth, Algebra is an efficient way of solving certain types of problems. Finally, it promotes intellectual activities such as generalizations, organized thinking, and deductive reasoning. Considering the above requirements and on the grounds of equity, no student should be denied access to learning algebra.

Stacey & Kendal (2004) listed each country's or its province's conception of what constitutes algebra. Algebra is conceived as : a way of expressing generality and pattern (British Columbia, England, Victoria, Singapore); a study of symbol manipulation and equation solving (Brazil, France, Germany, Hong Kong SAR, Hungary, Israel, Italy, Russian Federation); a way to solve problems beyond the reach of arithmetic (Czech Republic, France, Hungary, Italy, Japan, Hong Kong SAR, Singapore); a way to interpret the world through modeling (Quebec, England, Netherlands, Victoria); and a formal system dealing with set theory, logical operations, and other operations (Singapore, Hungary). Therefore, many countries have very common objectives of teaching algebra to their students.

In contrast, it is interesting to examine the syllabus and the teaching and learning of algebra in Sri Lanka. The syllabus is mainly conceived as a way of solving word problems, symbol manipulation and equation solving. Students start to learn algebra in grade 6, approximately at the age of 11. From there on, they study the subject as part of the school curriculum until the end of grade 13. There are no specific algebra courses that students can choose as electives within the school system. All high school students learn algebra as a component of their mathematics curriculum with other components such as geometry, statistics, probability or trigonometry.

The teaching of high school mathematics in Sri Lanka is still highly teacher-centred. A typical lesson starts with an explanation of concepts by the teacher, and students will then be given some exercises to do individually. Student numbers in classrooms are very high, leaving teachers with very few options to conduct group work or other activities or to use manipulatives. On the other hand, many students cannot afford to buy calculators or other forms of technology limiting further the teacher's options. Seat work, exposition, and chalk and talk are common in classroom teaching. In such a scenario, students are supposed to grasp the concepts by merely listening to the teacher and doing exercises with or without the teacher's help.

In recent years, many research projects on mathematics education have focused on learning difficulties of students related to algebra. Research suggests that solutions to the problem of student inability to be successful in algebra are many and frequently interconnected (Norton & Irvin, 2007). After studying several research findings, they suggested some solutions which include the following: making explicit algebraic thinking inherent in arithmetic in children's earlier learning (Lins & Kaput, 2004; Warren & Cooper, 2006), explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting (Kirshner & Awtry, 2004; Sleeman, 1986; Stacey & MacGregor, 1997, 1999; Stacey & Chick, 2004), especially in transformational activities (Kieran & Yerushalmy, 2004; Stacey & Chick, 2004), using multiple representations including the use of technology (Kieran & Yerushalmy, 2004; Van de Walle, 2006), and recognizing the importance of embedding algebra into contextual themes (National Council of Teachers of Mathematics, 1998; Stacey & Chick, 2004). Other research have also shown that student errors in algebra can be ascribed to fundamental differences between arithmetic and algebra. For instance, if students want to adopt an algebraic way of reasoning, they have to break away from certain arithmetical conventions and need to learn to deal with algebraic symbolism.

Algebraic problem solving could be considered in two ways. According to Kieran (1989), Sfard, (1991), and MacGregor (2004), these two aspects are "structural (systemic)" and "procedural (surface)". Sfard (1991) presented a similar idea saying that abstract mathematical notions can be conceived in two fundamentally different ways. They are structurally as objects and operationally as processes. She further added that the structural conception is static, instantaneous, and integrative while the operational conception is dynamic, sequential, and detailed.

This fundamental difference explains many reasons why students make a considerable number of errors while they are operating on the structural aspects of algebra. For instance, substituting various values for the variable in a simple equation until the correct value is found refers to the procedural aspect of algebra. Applying characteristics such as commutative or distributive laws or equivalency relationships to solve algebraic equations comes under the structural category.

In my view, students make many errors when they are dealing with the structural aspect of algebra. On the other hand, many algebraic problems are too hard for students, because solving them may require a secure understanding of the structural aspects of fractions, decimals, negative numbers, equivalence, ratios, percentages or rates. For example, students should understand that $\frac{1+3}{5}$ can be separated as $\frac{1}{5}+\frac{3}{5}$ in the same way they understand the reverse process. As Stacey & MacGregor (1997) pointed out, algebra learning should support the learning of arithmetic concepts instead of being taught as a separate topic. Arithmetic concepts could be utilized to reinforce the learning of algebra but, at the same time, these concepts could well hinder the same process as students tend to apply arithmetic concepts incorrectly to solve algebraic problems. In order to identify the structure of algebra and solve problems efficiently, students need to practice problems individually or in groups. This is happening in Sri Lankan classrooms but students still make many errors in the problem solving process. This paper presents the results of a study conducted in a sample of Sri Lankan high schools to find out student errors when solving algebraic problems and the root causes for these errors.

Method

The participants were 288 students (136 males and 152 females) in grades 9 and 10 in the Western province of Sri Lanka. First, a standardized algebra test with 30 items was administered to all students. The students were from a mixed-ability group in order to ensure the identification of a sufficient number of interviewees, since higher-ability groups would presumably contain fewer children making errors. A sample of 64 students was selected for interviews, which were organized within two weeks of the test. All care was taken to ensure that the sample included the largest possible variety of errors. Each child was interviewed individually and the results were tape-recorded. In the interview process, children were asked to redo the problem on a piece of paper and were asked for verbal explanations of their work whenever necessary. The interviewer followed the child's thought processes and asked spontaneously found questions within the interview framework.

The interview schedule adopted here was based on the method elucidated by Newman (1977), Casey (1978), and Clements (1980) which emphasizes an input, process and output model. Basically, the model says that when a child produces an incorrect answer to a question, the error resulting in that answer may have occurred at one of several stages in the process of solving that problem. The child may have misread the question (reading error), or may have misinterpreted it (interpretation error). Alternatively, despite a correct interpretation, an incorrect method may have been used to solve the problem (process error). Even though all the above steps are correct, the answer may have been wrongly encoded (encoding error). Still, the child may have a conflict with explaining or verifying the answer (verification error). There could, however, be other possibilities as well such as the possibility of any combination or interaction of the above errors. Also, there could possibly be psychological factors for making errors rather than mathematical factors such as low attention to the task, carelessness or lack of motivation.

The main questions in the interview schedule were categorized into eight sections. They were reading, comprehension, interpretation, strategy/skills selection, process, encoding, consolidation, and verification. The questions asked in each stage were: Please read the question, what does the question mean? How will you do the question? Work out the question and tell me what you are doing as you go? Write down the answer. What does the answer mean? Is there any way you can check to make sure that your answer is right? Finally, the interviewer asked some conflicting questions to verify whether the child has a conflict in the problem. As soon as the proper cause for the error was identified, the interview was terminated. The emphasis on analyzing student errors was based upon the Piagetian view that a consistently made error in a given problem reflects a way of viewing that problem or handling its solution, which is consonant with the child's cognitive structure. Therefore, analysis based upon the child's perspective and the way of functioning with respect to that task rather than upon the logic of the task, provides an insight into the child's cognition. For this purpose, the clinical interview procedure developed by Piaget and others was adopted in the interviews (Piaget & Inhelder, 1958). Each interview lasted within 15 to 30 minutes.

The interview transcripts in conjunction with children's written work were analyzed by the researcher seeking for patterns and regularities concerning the following questions: What is the percentage of incorrect responses for each item of the test? What error types are significant among the errors displayed by pupils? What procedures were used by them? Can we divide the errors into some characteristic groups? To what extent are there differences in errors made by males and females? What are the root causes of specific errors and misunderstandings?

Discussion of results

The greatest number of errors occurred during the process stage (57.8%) of the solving process followed by comprehension error (21.9%), encoding error (15.6%), and verification error (4.7%). None of the students had problems with reading the questions. There was no significant difference of making errors between males and females. Six prominent error types were identified among others and three of them are discussed below with their possible causes.

1. **Transformation of word problems into algebraic language (49.4%)**

Question: Amala is 4 years older than Nimala. The sum of their ages is 28 years. What is Amala's age?

Strategy used	**Possible reasoning for errors**
Amala's age = x Nimala's age = x + 4	Assigning letters to ages in the same order as they appear in the problem without considering the magnitudes
Amala's age = x Nimala's age = 4x	Same reasoning as above but considered 4 + x as 4x (conjoining)
Nimala's age = x Amala's age = 4x	Assigning letters in the correct order but conjoin 4 + x as 4x
Amala's age = x Nimala's age = y 4x – y = 0 x + y = 28	Same reasoning as above but uses two letters.
Amala's age = x Nimala's age = y x + 4 = y x + y = 28	Direct translation of the word problem into algebraic language without considering the magnitudes

Students often try to translate word by word or phrase by phrase from the problem into algebraic language causing wrong transformations. For example, students read the phrase "Amala is 4 years older than" and translate it as x + 4 while considering Amala's age to be x. Other possible reasoning is the poor understanding of the arithmetic-algebraic connection of a word problem, unwillingness to accept an algebraic expression in its original form (conjoining 4 + x as 4x), and an improper understanding of basic concepts which are needed to build an algebraic relationship.

2. **Parenthesis omitted (38.7%)**

Question: The length and the width of a rectangle were given as b + 2 and 5 respectively (a figure was provided). What is the area of the rectangle?

Strategy used	**Possible reasoning for errors**
b + 2 * 5 b + 10	Writing the answer without using parenthesis and simplifying the expression using "BEDMAS"
5 * b + 2 5b + 2	Same as above
b + 2 * 5 b + 10	Assuming b + 2 as one term but did not use brackets

(b + 2) * 5 5b + 10	Putting an extra multiplication sign with brackets (not understanding the syntax of algebra correctly)

Students often do not have a clear understanding of precedence of operations. Therefore, they usually do not see the need of brackets. When brackets are not present in an algebraic expression, students tend to simplify it either from left to right operating on immediate terms rather than considering the whole expression (the students who made this error were asked to read the expression 2 + 5x = 7x and the way they said it was 2 plus 5 (pause) x, conjoining was reduced to a certain extent when the expression was in the form 2x + 5), using BEDMAS rule or using any other way they choose as convenient. Most often, students concentrate on the immediate term next to an operation. Also, they face problems of perceiving numbers, signs and letters together as expressions and try to conjoin them to obtain one answer as they do in arithmetic problems (there is an interference with their arithmetic knowledge, for instance, 3 + 5*2 = 13 or adding of immediate like terms such as 2 and 5 in 2 + 5x to get 7x). An additional problem for them was the syntax of algebra such as incorrectly using brackets and multiplications together.

3. **Wrong operations in solving equations (29.2%)**

Question #1: $\frac{y-3}{-4} = 2y$

Strategy used	Possible reasoning for errors
3y - 4 = 2y	Considering 3 taken away from y as 3y and writing the denominator as part of the numerator
y – 3 = 8y	Ignoring the minus sign in cross multiplication
$\frac{y-2y}{-4} = 3$	Changing the sides of 2y and -3. (Directly operating on the variable without considering other quantities or operations)
$\frac{y-3}{-4} * -4 = 2y - 4$	Multiplying both sides by -4 but dropping the multiplication sign between 2y and -4
$\frac{y-3+3}{-4} = 2y + 3$	Directly operating on the numerator of an algebraic fraction without considering the denominator
$\frac{1-3}{-4} = 2y$	Arbitrarily substituting 1 for y
$\frac{y-3}{-2} = y$	Considering the equal sign as a multiplication and reducing the common factor of 2 and 4
-4 * y – (-4)(3) = 2y * -4	Doing two operations together incorrectly (cross multiplication and multiplying both sides by -4)

Question #2: If x = y + z and x + y + z = 30, then find the value of x.

Strategy used	Possible reasoning for errors
X = 30 / 3	Considering equation 2 only and x, y, and z to be

	the same
x + y + z = 30 5 + 10 + 15 = 30 x = 5	Assigning arbitrary values for letters considering their alphabetical order has ascending values
x + y + z = 30 x = 30	Arbitrarily assigning zeros to some variables
x + y + z = 30 x = 30 - y - z	Considering only the equation with numbers
$x = \frac{30}{y+z}$	Assuming a multiplication between x and y instead of an addition

Students often make poor strategic decisions when solving equations. When a single equation or two equations were given, the most common error types were assigning arbitrary convenient values such as 5, 10 or 15 for letters and operating on variables directly. They lack the ability to generate and perceive a global overview of different parts of an equation including the equal sign. Students often think that algebra is rule-based and attempt to apply rules by memorizing them without proper understanding of the structural aspect of the problem. Most students lacked the basic understanding of the equality sign, specific unknowns, and generalized numbers. For example, when two equations were given, they only consider the equation with numbers. The equations with letters are unacceptable to them and these equations make no sense to them. In addition to the above error types, there were a number of non-algebraic errors found in this study. Some of them were dropping a sign, incomplete workings, transcription errors, and other computational errors. Possible causes for some of these errors may be psychological factors rather than mathematical factors such as low attention to the task, carelessness or lack of motivation.

Conclusion and recommendations

Algebra is hard to teach and hard to learn. As MacGregor & Stacey (1997) have pointed out, difficulties in learning to use algebraic notation have several origins. Most of these difficulties appeared in the current study. The findings underline the importance of teachers having deep knowledge of mathematical content as well as insights into students' thinking. Teachers must identify the fundamental ideas that need to be taught, and understand the difficulties and misunderstandings that are likely to occur. Problems that can be solved using algebra have different structures and concepts. Using the same methods to teach every student may not always work. Some diagnostic teaching is sometimes necessary. It would be helpful if teachers could use the methods that have used in this research to identify student errors. When students are given opportunities to verbalize their mental processes, this would mostly facilitate the transfer from arithmetic to algebra. This would also help the teacher to identify specific student problems.

Another concern is the way that students have been given the opportunity to study algebra in the high school level. One algebra stream for all may not suffice. The questions that must be asked are "Is one algebra for all?", "What type of algebra?", and "How much for whom?" Students who are going to use algebra in higher mathematics need to develop a fluency and symbol sense that will let them operate with algebra as a language and as a set of problem-solving methods. Most of them should have relatively strong arithmetic and logical skills. These students need to be given a stronger algebra diet and they should be encouraged to make conceptual jumps to algebraic thinking. In my view, they should have a thorough understanding of the structural aspect of algebra. Those who have weak arithmetic skills should strengthen their understanding of number and their ability to solve problems by logical reasoning. The emphasis here should be on the procedural aspect of the subject. Even by systematically guessing answers should not be considered as trivial at this stage.

References

Casey, D. P. (1978). Failing students: A strategy of error analysis. In P. Costello (Ed.), *Aspects of motivation* (pp. 295-306), Mathematical Association of Victoria.

Clements, M. A. (1980). Analyzing children's errors on written mathematical tasks, *Educational studies in Mathematics*, *11*, 1-21.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp 33-35), NCTM.

MacGregor, M. (2004). Goals and content of an algebra curriculum for the compulsory years of schooling. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra, The 12th ICMI Study* (pp. 313-328), Kluwer Academic Publishers.

MacGregor, M. & Stacey, K. (1997). Students understanding of algebraic notation: 11-15, *Educational Studies in Mathematics*, *33*, 1-19.

Newman, M. A. (1977). *An analysis of sixth-grade pupil's errors in written mathematical tasks*, Research in Mathematics Education in Australia, Swinburne Press.

Norton, S. & Irvin, J. (2007). A concrete approach to teaching symbolic algebra. In J. Watson & K. Beswick (Eds.) *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*, MERGA Inc. Retrieved: Dec. 20, 2007 from http://www.merga.net.au/documents/RP502007.pdf

Piaget, J. & Inhelder, B. (1958). *The growth of logical thinking from childhood to adolescence*, Routledge & Kegan Paul Ltd.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics, 22*, 1-36.

Stacey, K. & Kendal, M. (2004). Algebra: A world of difference. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra, The 12th ICMI Study* (pp. 329-346), Kluwer Academic Publishers.

Stacey, K. & MacGregor, M. (1997). Building foundations for algebra. *Mathematics Teaching in the Middle School, 2(4)*, 252-260.