IMPROVING ADVANCED STUDENTS' PROVING ABILITIES

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Abstract. In this paper, we describe preliminary results arising from the development of a modified R. L. Moore Method course devoted entirely to helping advanced university mathematics students improve their proving abilities. The paper describes the course and why it might be needed. We also discuss kinds and aspects of proofs in a way that may be useful in gauging student progress, and in particular, introduce the idea of the formal-rhetorical and problem-centered parts of a proof. We go on to propose a theoretical perspective suggesting that much of proving depends on procedural knowledge in the form of small habits of mind, or behavioral schemas, and give three examples.

This paper reports on results arising from an ongoing design experiment. The purpose of the experiment is to examine ways advanced undergraduate and beginning graduate mathematics students construct, and learn to construct, proofs and to design a course to facilitate that learning. The paper has a research orientation, but the research is based directly on practice, that is, it is based on observations the authors made during the early development and teaching of the course. Also, the results should be directly applicable to such teaching. The theme of the paper concerns both cognition, in particular, the proving process, and teaching, in particular, helping students learn to prove theorems. The paper has three main parts: a description of the design experiment, including the course; a discussion of kinds and aspects of proofs, needed to assess students' progress; and a discussion of students' abilities, actions, and difficulties in the proving process.

1.THE DESIGN EXPERIMENT

The design experiment is organized around the development and teaching of a one-semester course whose sole purpose is to help advanced university mathematics students improve their theorem-proving abilities. That is, the course is concerned with differing kinds and aspects of proofs, and proving, rather than with conceptual understanding of a particular mathematical topic, such as abstract algebra or real analysis. We are reporting on the first two, of a projected eight, iterations in the development process. The course was preceded by three pilot versions, during which data was only informally collected. We expect that the findings of this study will be useful in teaching, not only advanced university students, but also secondary school and beginning university students.

1.1 The course description

The course carries three credits and meets for 75 minutes twice per week for one semester, making a total of about 30 class meetings. The teaching is a modification of the R. L. Moore

Method (Jones, 1977; Mahavier, 1999). That is, instead of a book, students are given notes, including statements of theorems, definitions, and requests for examples, but no proofs, and only minimal explanations. There are no formal lectures, and students construct proofs at home and present them in class.

The experience of the first two authors suggests that students who succeed in Moore Method courses develop an unusually deep understanding of the mathematical content and also considerably improve their ability to construct proofs. However, some students may not succeed at all. Thus our modification consists in focusing on proving in general, rather than on a particular mathematical topic, and in offering considerable in-class advice on improvements to attempted proofs. We also offer tutoring for a few students.

All three authors attend class, with the third author video recording presentations and discussions and taking field notes. Between class meetings, we meet for about an hour to analyze the previous class meeting and plan how we will try to enhance the students' learning in the next meeting. These analysis and planning sessions are also video recorded, and all of this data is later re-analyzed in planning the next iteration of the course. The Fall Semester version of the course covers some basic ideas about sets, functions, real analysis, and semigroups. The Spring Semester version covers sets, functions, some real analysis, and topology. The specific topics covered are of less importance than giving students opportunities to experience as many different types of proofs as possible. The theorems in the two versions do not overlap, and students can enroll in either or both versions of the course.

1.2 Philosophical and theoretical perspectives

We are taking a constructivist approach in that we assume students will construct their own theorem-proving abilities through attempting to write proofs. Thus the course seeks to maximize students' proof writing opportunities. However, in our criticism and advice, we take a more Vygotskian approach, in that we are representing the wider mathematical community in describing the way proofs are written, that is, in describing the genre of proof. Of course, this assumes there is such a thing as a genre of proof and that proofs are not just convincing arguments, but those written in a special way. This seems not to be universally accepted among mathematics education researchers, but it appears to help our students write proofs, especially for statements they see as obvious.

Finally, we see (at least the conscious part of) cognition in general, and the proving process in particular, as a sequence of mental and physical actions, such as writing or thinking a line in a proof, drawing or visualizing a diagram, reflecting on the results of earlier actions, or trying to remember an example. Many such actions appear to be guided by small "habits of mind" that often link a particular recognized situation to a particular action. Such habits of mind reduce the burden on working memory, and teachers can encourage their development.

1.3 The students

Almost all students in our courses were advanced undergraduate or beginning graduate mathematics students at a mid-size Ph.D.-granting state university in the U.S.A. However, one student had passed the qualifying examinations for the Ph.D. In general, students selfselected the course, but a few were advised to take it because they were having difficulties in other mathematics courses.

1.4 The need for the course

Although most graduate and advanced undergraduate mathematics courses are concerned with the concepts, examples, and relationships among concepts within some topic such as real analysis, the students are expected to demonstrate their conceptual understanding by writing original proofs. Many students have difficulty constructing such proofs even if they understand the relevant concepts. For example, at the beginning of the two versions of our course that we have now taught, most students could not prove that *if f and g are functions from A to A and $f \circ g$ is one-to-one, then so is g.*

2. KINDS AND ASPECTS OF PROOFS

Normally a university mathematics course is associated with an implicit theoretical framework that allows the progress in students' knowledge, understanding, and abilities to be discussed or reflected upon. For example, in an undergraduate abstract algebra course a student might, or might not, understand key theorems, such as the isomorphism theorems, concepts, such as homomorphism or coset, and examples such as Z_3. The course we are developing requires a similar framework, but one about kinds and aspects of proofs and associated student abilities. In addition to considering the familiar direct, indirect, and induction proofs, we have found the following two distinctions useful.

2.1 The formal-rhetorical and problem-oriented parts of a proof

The *formal-rhetorical* part of a proof is the part that can be written depending only on the formal, logical aspects of definitions and theorems without recourse to their deeper meanings or to problem solving in the sense of Schoenfeld (1985, p. 74). The remaining *problem-oriented* part of a proof does depend on problem solving and a deeper understanding of the concepts involved (Selden & Selden, in press). Within the problem-centered part of a proof, it might be useful to contrast subproofs that are syntactic or semantic in the sense of Weber and Alcock (2004), but we have not investigated this.

An example of the formal-rhetorical part of a proof can be seen in a proof of Theorem 22 in our Fall 2007 notes: *If f and g are real valued functions of a real variable continuous at a, then $f + g$ is continuous at a.* The formal-rhetorical part of a typical proof of this theorem is as follows.

Let f and g be functions and suppose they are continuous at a. Suppose ε is a number > 0. Let δ = _____ . Note that $\delta > 0$. Let x be a number. Suppose that $| x - a | < \delta$. _____. Thus $| f(x) + g(x) - (f(a) + g(a)) | < \varepsilon$. Therefore $f + g$ is continuous at a.

The problem-oriented part of the proof consists of filling in the two blanks using minimum in the first and the triangle inequality in the second.

In the Spring 2008 course, we began with an early emphasis on the formal-rhetorical part of proving. Few students came to us able to write this part of proofs, but our experience suggests that we can teach all students to do so. Writing the formal-rhetorical part of a proof appears to alleviate a major student difficulty, namely, focusing on using the hypotheses, before focusing on what is really to be proved. Also, we hope an early emphasis on the formal-rhetorical parts of proofs may allow students to build what one might call "success momentum," preventing them from reducing their efforts or quit work entirely, due to lack of success. The next section speaks to the problem-oriented part of a proof .

2.2 Proofs requiring a previous result

In relating proofs to the notes, we have begun distinguishing three kinds of proofs beyond those following immediately from definitions: (1) those requiring a result that is in the notes; (2) those requiring a result that is not in the notes, but is easily articulated; and (3) those requiring a result that is not in the notes, and is not easily articulated. There are students who fail to write a proof of type (1) because they do not look back at their notes to find a needed theorem. For example, this occurred in Fall 2007 for Theorem 24, which states that polynomials of any degree are continuous. Theorems 19-23 state that sums and products of continuous functions are continuous and that the identity and constant functions are continuous. This is enough to prove Theorem 24 by induction, but there were students who did not notice this and could not prove it.

For an example of a type 3 proof we turn to the Fall 2007 notes where a semigroup is defined to be a nonempty set S together with an associative binary operation, and an ideal I of S is a nonempty subset so that $IS \cup SI \subseteq I$. Theorem 46 says that *if S is a commutative semigroup with no proper ideals, then S is a group.* The first result needed is that if $a \in S$, then aS is an ideal (and hence $aS = S$). The second result needed is that if $aS = S$, then the equation $ax = b$ can be solved for x. These results were not included in the notes in order that students could experience a type 3 proof.

3. ACTIONS, ABILITIES, AND DIFFICULTIES IN THE PROVING PROCESS

We start this section with an example of a small part of a proof construction that we see in terms of a situation and a mental or physical action. Most mathematicians would regard the action as routine and the example as unexceptional, but its very unexceptional nature suggests that many proofs constructed by students consist largely, although not entirely, of such <situation, action> pairs.

Mary, an advanced graduate student in mathematics, described to us a situation and a resulting mental action while proving that a compact set A in R^n is bounded. Mary and two graduate student collaborators assumed A to be unbounded and were able to construct an open cover of A that had no finite subcover. They then immediately observed that this contradicted the compactness of A and proved the result. Mary, who had never studied formal logic, reported that, upon finding the cover had no finite subcover, she immediately knew the result was proved. She did not reflect on the logical structure of what had transpired in an effort to explicitly warrant that the proof was complete. Mary said her two collaborators also did not appear to require reflection or an explicit warrant.

We infer that each student recognized a situation involving a hypothesis (the compactness of A), a conclusion assumed false (the unboundness of A), and a resulting contradiction (the noncompactness of A). The mental action was simply deciding the result had been proved. The students' linking of the situation to the action appeared to be automatic and not to require reflection or a warrant (such as one based on logic). We see this as due to the students' extensive proof constructing experience. Many less experienced students would require considerable reflection and wonder needlessly what should be contradicted.

As a person gains experience, much of proof construction appears to be separable into sequences of small parts, consisting of recognizing a situation and taking a mental or physical action. Actions which once may have required a conscious warrant become automatically linked to triggering situations. From a third-person, or outside, perspective these consistently linked <situation, action> pairs might be regarded as small "habits of mind" (Margolis, 1993). Taking a first-person, inside, or psychological perspective, they are lasting mental structures that we have called "behavioral schemas." Enacting them depends on conscious awareness of a situation, but does not require conscious processing before the action. Such schemas are immediately available, that is, do not have to be recalled before use (Selden & Selden, 2008).

We suggest that behavioral schemas are a form of procedural knowledge. They can be acquired or learned through repeated similar proof writing experiences, and their enactment reduces the burden on a prover's working memory. Some schemas contribute to constructing correct proofs and might be regarded as abilities. Others may block proof construction and engender recurring difficulties. Here are some examples.

3.1 Proving universally quantified statements

One often starts the proof of a statement containing a universally quantified variable, such as "For all (numbers) x P(x)" by writing, "Let x be a number," meaning x is "fixed, but arbitrary." Our students appear to eventually come to do this in a way that suggests they are enacting a behavioral schema. However, some students are initially reluctant to write "Let x be a number" in their arguments. Thus, constructing the schema and becoming comfortable with it can take considerable time.

We illustrate this by drawing on an interview with Mary, the advanced graduate student mentioned above. In the interview, she described her experiences in Dr. K's beginning graduate real analysis course, taken two years earlier. In that course, Dr. K assigned three or four weekly proofs, graded them very thoroughly, and allowed them to be resubmitted for regrading. He emphasized writing things like "let x be a number" into proofs, and this is called for in many real analysis proofs. Dr. K, whom we also interviewed, agrees with Mary's description of the course, and recalls that her homework grades steadily improved. Mary recalls clearly not feeling that the requirement to write things like "let x be a number" was particularly important or appropriate. She did so, however, because "well, he said to do it, so that's how I am going to do it, because I want to get full credit." Dr. K thought Mary followed his advice because she wanted a good grade. Mary recalls that, near the middle of the course, she came to feel that writing things like "let x be a number" into proofs "made sense and it was the way to do it." She reports that no single memorable event occurred that contributed to this feeling. Mary also said that now, two years later, she cannot think of any other way to write (this aspect of) proofs.

A student coming to regard an action as appropriate or correct only after doing it many times, partly for non-mathematical reasons, may seem somewhat counterintuitive to some researchers in mathematics education. It is, however, in accord with work in philosophy on the acquisition of a "feeling of rightness" (James, 1890; Mangan, 2001). However long this behavioral schema takes to develop, once developed, it might be viewed as a proof construction ability. In contrast, the next example of a behavioral schema, when enacted, will often produce a difficulty.

3.2 Focusing too soon on the hypotheses

In a paper reporting on a mid-level undergraduate transition-to-proof course, R. C. Moore (1994) described students' attempts to prove: *Let f and g be functions on A. If $f \circ g$ is one-to-one, then g is one-to-one.* This was a final examination question and "all but one student [of 16] incorrectly attempted to begin [the problem-solving aspect of] the proof with the hypothesis – $f \circ g$ is one-to-one – rather than [starting to prove the conclusion] by assuming $g(x) = g(y)$ for a fixed but arbitrary x and y in A."

We have noted a similar, persistent difficulty when students who were unable to construct a complete proof were asked to provide as much of one as they could. After writing little more than the hypotheses, some students turned immediately to focusing on those hypotheses, after which they could not complete the proof.

For example, on the twenty-sixth day of the Spring 2008 course, Willy was asked to prove Theorem 29: *Let X and Y be topological spaces and $f : X \to Y$ be a homeomorphism of X onto Y. If X is a Hausdorff space, then so is Y.* Because only ten minutes of class time remained, we asked Willy to "do the set-up," that is, present the formal-rhetorical part of the proof. Willy had indicated that he had not yet proved this theorem. However, early on in the course, he had developed some ability to write the formal-rhetorical parts of simple proofs.

On the left side of the board, Willy wrote: "Proof. Let X and Y be topological spaces. Let $f : X \to Y$ be a homeomorphism of X onto Y. Suppose X is a Hausdorff space. __________ . Then Y is a Hausdorff space." Then, on the right side of the board, he listed, "homeomorphism: one-to-one, onto, continuous (f is open mapping)" and then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, "And, I was just trying to just think, homeomorphism means one-to-one, onto, …" After some discussion about the meaning of homeomorphism, the first author said, "There is no harm in analyzing what stuff you might want to use, but there is more to do before you can use any of that stuff." Willy did not make further progress that day.

We inferred that Willy was enacting a behavioral schema in which the situation was having written little more than the hypotheses, and the action was focusing on the meaning and potential uses of those hypotheses before examining the conclusion. This schema often leads to difficulties, as it did for Willy.

We conjectured that, had Willy not been distracted by focusing on the meaning of homeomorphism, he might have written more of the formal-rhetorical part of the proof. That is, he might have filled in the blank space of his proof with something like, "Let y_1 and y_2 be two elements of Y. ________ . Thus there are disjoint open sets U and V so that $y_1 \in U$ and $y_2 \in V$." Writing this exposes the "real problem" in this theorem, which Willy might have found tractable. Indeed, by the next class meeting, he had constructed a proof in the way we had expected.

3.3 Showing an object is in a set

Here is an example of a tutor, the first author, leading a student, Sofia, towards constructing a behavioral schema. The hour and a half session occurred in the middle of the Spring 2008 course, and was devoted to helping Sofia prove Theorem 20: *Let* $(X, \mathcal{U})$ *be a topological space and* $Y \subseteq X$. *Then* $(Y, \{U \cap Y \mid U \in \mathcal{U}\})$ *is a topological space (called the relative topology on* Y*).*

Sofia indicated she did not know how to prove the theorem, but at the tutor's suggestion, she wrote the first and last lines of a proof and drew a sketch. With guidance, she then unpacked what was to be proved into four parts (the four defining properties of a topology), and she proved the first one, that is, $Y \in \{U \cap Y \mid U \in \mathcal{U}\}$. The tutor then tried to encourage her reflection on a key point, saying, "Now, if I didn't understand that, and I asked you why Y is in there? The thing you said up above [$X \cap Y \in \{U \cap Y|U \in \mathcal{U}\}$] doesn't have any Y in it. You would have to tell me that, well, Y is equal to $X \cap Y$, OK no problem." Sofia said, "um hum."

Then Sofia wrote, "(2) Since [the empty set] $\varnothing \in \mathcal{U}$" and after 30 seconds said, "I'm stuck." It became clear that she did not know, in general, how to show an object is in a set when the defining variable in the set is compound, (for example, $U \cap Y$). The tutor then said to forget the theorem for a moment and turn to the natural numbers. He wrote $\{2n \mid n \in N\}$ and asked "Is 6 an element of that set?" Sofia answered yes, and the tutor asked why. Sofia and the tutor then agreed that the answer was yes because 6 = 2 x 3 and $3 \in N$. The tutor reiterated that 6 had to be represented in the form $2n$, where n had the appropriate property, that is, $n \in N$. Using this as a model, Sofia was then able to show $\varnothing \in \{U \cap Y \mid U \in \mathcal{U}\}$. With guidance, but less guidance than before, she also proved the third and fourth parts of the proof.

Here, the tutor's guidance and suggestions, not only helped Sofia prove the theorem, they also facilitated the construction of a behavioral schema in which the situation is needing to show an object is in a set (where the defining variable is compound), and the mental action is forming the intention to represent the object in the proper form and show the defining property of the set is satisfied. Sofia's previous work in the course suggested she already understood the language of sets and could draw appropriate Venn diagrams. However, for her that understanding was not equivalent to knowing how to use her knowledge in constructing this proof.

4. CONCLUSION

In this paper we described a design experiment to develop a course whose sole purpose is to improve advanced mathematics students' proving skills. The course appears to be useful because many advanced students have difficulty constructing proofs, and students' proofs are used as a major component in assessments of their understanding of content courses, such as modern algera or real analysis. The extent of advanced students' proving difficulties and the use of proofs for assessing students' understanding could be further investigated.

A theoretical framework for analysing student progress in such courses is beginning to emerge. We suggested it was useful to distinguish between the formal-rhetorical part of a proof and the problem-centered part. We also distinguished three levels of proof depending on whether the proof used results that were, or were not, in the notes, and the difficulty of articulating needed results not in the notes. We expect that additional distinctions would be useful in keeping track of students' progress.

We also introduced a theoretical perspective suggesting that much of a proof depends on procedural knowledge in the form of small habits of mind or behavioral schemas. We gave an example of such a schema that might be seen as a proving ability and another that tended to produce a proving difficulty. Finally, we gave an example of a tutor helping a student to construct a beneficial schema and noted that that schema was not equivalent to the student's relevant conceptual knowledge. We suspect there is much more to be investigated about behavioral schemas, the way they interact, and how they are learned.

REFERENCES

James, W. (1890). *The principles of psychology*. New York: Holt.

Jones, F. B. (1977). The Moore Method. *American Mathematical Monthly, 84*(4), 273-278.

Mahavier, W. S. (1999). What is the Moore method? *PRIMUS, 9,* 339-354.

Mangan, B. (2001, October). Sensation's ghost: The non-sensory "fringe" of consciousness. *Psyche*, 7(18). Retrieved 10/15/1007, from http://psyche.cs.monash.edu.au/v7/psyche-7-18-mangan.html .

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Margolis, H. (1993). *Paradigms and barriers: How habits of mind govern scientific belief* University of Chicago Press: Chicago.

Selden, J., & Selden, A. (in press). Teaching proving by coordinating aspects of proofs with students' abilities. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *The learning and teaching of proof across the grades.* London: Lawrence Erlbaum Associates/Taylor & Francis.

Selden, J., & Selden, A. (2008). Consciousness in enacting procedural knowledge. *Proceedings Conference on Research in Undergraduate Mathematics Education.* http://cresmet.asu.edu/crume2008/Proceedings/Proceedings.html .

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, 56(3), 209-234.