Didactical Reflections on the teaching of mathematical modelling

-Suggestions from concepts of "time" and "place" -

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The characteristics of mathematics in a society seem to differ according to the "time" and "place" which the mathematical model has constructed. This paper discusses that concepts about "time" and "place" have an important role in the practical teaching and learning of applications and modelling, for the following three aspects: (1) Problem situations which people are interested in, (2) Purposes for using mathematics in a society, (3) Existence of mathematical models, methods, etc. embedded in a society.

1. Introduction

When we consider the teaching and learning of mathematical modelling in school mathematics, it is meaningful to analyze the characteristics of mathematics outside of school, namely in a society. It is noted by Niss(2008) that any application of mathematics relies on the introduction of a mathematical model, whether explicitly or implicitly. Therefore, any mathematical model has to be constructed by someone. Then, when and where was each mathematical model constructed by someone? The characteristics of mathematics in a society seem differ according to the "time" and "place" which the mathematical model has constructed. It is suggested that concepts about "time" and "place" have an important role in the teaching and learning of applications and modelling. The following three points are discussed, that differ according to "time" and "place".

- Problem situations which people are interested in
- Purposes for using mathematics in a society
- Existence of mathematical models, methods, etc. embedded in a society

2. Problem situations which people are interested in

Concerning "place", problem situations which people are interested in differ according to the place where people are living, such as country, area, etc. It is obvious that the problem situations people face or are interested in differ between developing countries and developed countries. Further, even in the same country, problem situations also differ between urban areas and suburbs.

For example, the following problem situation is relatively familiar for students living in an oil producing country. On the other hand, it is not familiar for students living in a country which does not produce oil.

> A pipeline is to be constructed connecting an oil rig at sea to a nearby refinery on the coast. Find the cheapest method of constructing a pipeline taking into account the difference in cost

> of construction for 'on land' versus 'under water' sections of the pipeline. The 'under water' cost can be expected to be greater than the 'on land' cost. Because of the difference in cost of 'on land' and 'under water' sections, the cheapest cost will not automatically correspond to the shortest length of pipeline. (Board of Studies, 1997)

Concerning "time", it is also said that problem situations which people are interested in differ between past society and present society. For example, constructing a figure to measure the length or angle was important in past society, but as we now have convenient instruments to measure, it is not important at present.

Further, as Jablonka (2007) noted, "different purposes may result in different mathematical models of the 'same' reality." An example is explained as follows. "For the problem of a bank employee (aided by a software package), who must advise a client in the comparison of financing offers for a mortgage, for the manager of the bank this is a problem of profitability, and for the customer it is one of planning her personal finance."

Before thinking about the teaching and learning of applications and modelling, it is suggested for the teacher to understand the differences of problem situations according to the place, time, and person. Further, the teacher should understand that problem situations for people in a society and problem situations for students are also different. The teacher should select an appropriate modelling task so that students can make sense from the problem situation proposed by the teacher. This suggestion raised practical questions, such as "What is an appropriate modelling task?" Galbraith (2007) notes the following two meaningful points regarding this question.

(T1) Consistency with avowed purpose

"For example, if applications and modelling is included in mathematics education to attain goals such as 'students will experience school mathematics as useful for solving problems in real life outside the classroom' then students, to some extent, need to encounter tasks that are close parallels to comparable problem situations encountered outside the mathematics classroom."

(T2) Introducing real world modelling tasks

The following two aspects are raised; "(a) the importance of using models based on experience (influenced by student background)", "(b) motivation e.g. looking to the world and other disciplines for knowledge and problems."

(T1) is a basic important issue which may be neglected by some teachers in practical teaching. Regarding (T2), more detail discussion is required regarding practical teaching. For example, the following points should be discussed regarding (T2)(a).

T2 (a1): Does the problem situation concern the surroundings of students at present, or in the future?

(a2): If the problem situation concerns the present surroundings of students, is it concerned with most students or a few students? For example, mirror problems and "rock-paper-scissors" problems etc. are familiar with most Japanese students. But problems about soccer, tennis, rugby, etc. are only familiar with students who are interested in these sports.

(a3): If the problem situation concerns the surroundings of students in the future, is it concerned with the situation confronted as citizens, as individuals or in their profession/vocation? The former two situations concern many students. But profession/vocation situations only concern the particular students who want to work in that direction.

Regarding (T2)(b), it was suggested that students' motivations should be considered. However, we must notice that students' motivations differ according to each students' interests and experience. The following two points should be considered. First is the reason why someone had to solve the problem. This concerns the background or goals for developing the mathematical model. It is not clear for others why someone had to solve the particular problem. As a result, it becomes very hard for others to develop the mathematical model. Therefore, students should know the reasons before solving a real world problem. For example, real world problems about fitting functions to the given data are often used in teaching and learning modelling. When Japanese forecast when cherry trees will bloom, the reason for fitting the functions to the given data is obvious for the particular people, because they need to determine the date beforehand to do a party on the weekend for watching the cherry blossoms. However, when we forecast the cooling rate of coffee, what are the reasons for fitting the function to the given data? In general, a mathematical model is developed to attain a particular purpose. It is meaningless to fit a function without a purpose. In the case of the cooling rate of coffee, it requires a problematic situation so that students can derive questions like "How does the temperature decrease as time passes?" and "Is it proportional or not?"

Second, it is important for the teacher to evaluate whether or not students can accept the problem posed by someone as their own problem. If students can accept the problem posed by someone as their own problem, it is an ideal setting for the teacher to treat modelling activity in the classroom. One of the methods is for the teacher to let students select an interesting task among several modelling tasks presented by teacher, and tackle it individually or in a group. Otherwise, the teacher proposes a series of observations or actions, so that students can use these to derive similar problems. This approach begins by observing or analyzing the phenomenon or action by students. Students are expected to derive key questions which will be solved by using mathematics.

For example, the teacher asks students "What shapes of cans are used in a supermarket? Let's examine them this weekend." Through this activity, students find out that most cans are cylinders, though some are not cylinders. Further, considering the relationships between the shapes and contents of cans, they will find that there are two types. In the first type, the shape is affected by the can's contents, while the second type is not affected by the can's contents. Then students find that the shapes of cans which are not affected by the contents are generally cylinders. Through these activities, students gradually formulate the real problem like "Why are the shapes of cans generally cylinders, not cubes?"

Let's pick one more example, the problem about baton passing in a relay (Osawa, 2004). Students discuss how they can win in a relay in school sports. Several issues are derived from students to win a relay, such as the order of runners, how to pass the baton, etc. When focusing on the baton pass, the following problem is formulated in verbal terms "When does the next runner begin to run to get the baton from the previous runner, for the shortest baton pass time? In other words, what is the best distance between the previous runner and the next runner, so the next runner can get the baton with the least time loss?" In this example, the problem is derived by students themselves. The reason why they need to solve this problem is obvious for students.

For teaching modelling, it is crucial to propose appropriate observations or actions involving discussion between the teacher and students, so the students can accept the proposed problem as their own.

3. Purposes for using mathematics in a society

The purposes for using mathematics differ according the place people are working on. Niss (2008) identified three different kinds of purposes for using mathematics in other disciplines or areas of practice:

(P1) in order to *understand* (represent, explain, predict) parts of the world,

(P2) in order to subject parts of the world to some kind of *action* (including making decisions, solving problems),

(P3) in order to *design* parts or aspects of the extra-mathematical world (creating or shaping artifacts, i.e. objects, systems, structures).

I think these three purposes are important for the following three points.

(1) Educational goals students are expected to attain

(2) Understanding of modelling process for the beginner

(3) Appreciation of the usefulness of mathematics in society

(1) Educational goals for what students are expected to acquire

This first point is characterized by the question "What kinds of educational goals are emphasized in teaching and learning mathematical modelling?" Modelling is treated for a variety of educational goals, such as foundations of science, critical citizenship, professional/vocational preparation, way of living, etc. There seems to be a strong connection between purposes for using mathematics and educational goals. So, what are the relationships between the three purposes and educational goals?

In the case of "(P1) *understand*", parts of the world are considered to be phenomenon of extra-mathematical domains such as natural or social phenomenon. The mathematical model is verified by contrasting it with real data taken from the phenomenon. Therefore, the aims such as foundation of science, professional/vocational preparation are emphasized more when we treat mathematical models which aim to "understand". In the case of "(P2) *action*", parts of the world are considered problem situations, in which people have to make a decision or solve a problem. Two types of mathematical model seem to exist. First is a social system model which is developed to make an objective and safe decision among people in a society, such as taxi prices or railway schedules. These models concern all citizens. After this mathematical model is embedded in a society, it becomes a main source for the reconstruction of reality (Skovsmose, 1994). The second is developed with personal purposes, such as planning a family trip, or planning for family savings or loans. However, we must again note that "different purposes may result in different mathematical models of the "same" reality (Jablonka, 2007)." For example, trip planning may become part of a tour conductor's job. The mathematical model developed is effectively validated by developing another model to compare it with. Therefore, aims such as critical preparation for citizens and professions/vocations are emphasized more when we treat mathematical models which have the purpose of "action". In the case of "(P3) *design*", the focus is on objects which make our life more comfortable, such as furniture, architecture and designs using tessellation. This type of object is evaluated by individual sense of value. Therefore, the aim of professional/vocational preparation is emphasized more when we treat a mathematical model which has a purpose to "design". When we consider the teaching of modelling, we should examine the relation between the purpose for using mathematics and educational goals.

(2) Understanding of modelling process for the beginner

Second, these three purposes imply that the modelling process depends on the purpose or area of other disciplines. For example, when we "understand" the natural/social phenomenon, the mathematical model is abstracted from the real world phenomenon, and the mathematical model is verified by contrasting it with the real world phenomena. However, when we make an "action" or "design", multiple mathematical models are developed to make a decision, and the appropriate mathematical model is selected among several models according to the aim.

When we introduce mathematical modelling for students, a particular diagram of modelling process is often used to let students understand roughly what modelling is. We have to pay more attention to the fact that the modelling process differs according to the purpose for using the mathematics or area of other disciplines, and we need to consider the reason why the teacher introduces the particular modelling diagram for students.

(3) Appreciation of the usefulness of mathematics in society

Third, these three kinds of purposes are also useful when we teach the usefulness of mathematics to students. When we teach how mathematics is used in a real world situation, one of the methods is to identify purposes for using mathematics in the real world. By tackling a series of modelling tasks, students are expected to reflect and find out the purposes for using mathematics in a variety of cases studied before. For example, one of the methods is for the teacher to assess students' appreciation of the usefulness of mathematics by asking "How is mathematics useful when we see real world situations from a variety of viewpoints?" before and after modelling teaching. The teacher can assess how students deepened their appreciation of the usefulness of mathematics in a society, by comparing their writing before and after teaching modelling. For example, regarding the criteria to assess students' writing, one of the following criteria can be set as the viewpoint:

- Criteria 1: From only students' personal perspectives, not from social perspectives,
- Criteria 2: From social perspectives, which are not clear or only special cases,
- Criteria 3: From social perspectives which are clear and integrated, such as the three different kinds of purposes identified by Niss.

4. Existence of mathematical models, methods, etc. embedded in a society

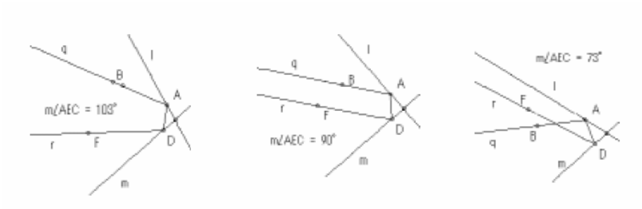
The last aspect is existence of mathematical models, methods, concepts, etc. embedded in daily life or society. The quantities and qualities of mathematical models, methods, concepts, etc. embedded in daily life or society differ between past and present society. Niss (2008) noted that in the past, the mathematical methods which we used were searched for by building up a mathematical model, but at present the mathematical methods are used with cultural techniques.

Further, as Skovsmose (1994) noted, “Mathematics not only creates ways of describing and handling problems, it also becomes a main source for the reconstruction of reality.” In particular, we need to pay attention to the existence of a variety of matters which people have developed to solve real world problems in the past, which are familiar and embedded in daily life or society at present, such as objects, systems, and structures which were built through full modelling processes in the past (e.g. taxi prices, railway schedules, savings and loans, bicycle reflectors, suspension bridges, methods to find and earthquake location, etc.).

When we think about the teaching and learning of applications and modelling, we need to first pay attention to the fact existence of a variety of matters familiar to students, which were developed using mathematics. This fact leads to two quite different approaches to teaching mathematical modelling. The first approach is to present to students a matter which someone else has developed in the past, and let students interpret it. The following is the problem about a bicycle reflector, which is one of the cases regarding objects which were built through a modelling process in past, and which are embedded in daily life at present.

> The reflector of a bicycle reflects light brightly. If we conduct an experiment in a dark room using an object from daily life like a penlight, it seems that the bicycle reflector reflects light in the direction exactly opposite to which it came from. What kind of structure does it have?

In this problem, by observing the structure of a reflector, students are expected to find that the angles with which each mirror is constructed are right angles. Therefore, the reason why the angles are right angles is investigated. This means that students interpret the reason why someone has made the mirror angles to be right angles. As the situation is 3-dimensional, it is difficult for students to consider. Students are expected to consider the simple case, namely a 2-dimensional situation as follows. Students are expected to find out and prove the fact “When the angle between two mirrors is 90 degrees, the incident light and reflected light are almost parallel”.



Then, students consider the 3-dimensional situation. Like this example, when the object or system developed to solve a real world problem in the past and which is embedded in daily life or society at present is considered in the classroom, we expect the following teaching flow for students to follow.

(1) To clarify why and for whom the object or system has been developed.

(2) To find out the characteristics of the object or system by observing and analyzing it.

(3) Considering the intention or role why the object or system is used in a dairy life or society, the mathematical structure of the characteristics found in (2) is investigated. Mathematical analysis is done in this stage.

(4) Appreciating the wisdom of ancestors. Then, by analyzing implicit assumptions established

in this situation, further modifications are considered.

This teaching flow is different from the full modelling process which is generally recognized. The following is the full modelling process by Pollak (1997).

(1) We identify something we want to know, do, or understand. The result is a question in the real world.
(2) We select “objects” that seem important in the real-world question and study the relations among them. The result is the identification of key concepts.
(3) We decide what we will examine and what we will ignore about the objects and their interrelations. You simply cannot take everything into account. The result is an idealized version of the original question.
(4) We translate this idealized version into mathematical terms, and obtain a mathematical formulation of it.
(5) We identify the field(s) of mathematics that are needed, and bring to bear the instincts and knowledge of those fields.
(6) We use mathematical methods and insight, and get results. Out of this step come techniques, interesting examples, solutions, theorems, algorithms.
(7) We translate back to the original field and obtain a theory of the idealized question.
(8) Now comes the reality back. Do we believe what is being said? Are the results practical, the answers reasonable, the consequences acceptable?
(a) If yes, the real –world problem solving has been successful, and our next job-both difficult and extraordinarily important- is to communicate with potential users.
(b) If no, we go back to the beginning. Why are the results impractical, or the answers unreasonable, or the consequences unacceptable? Because the model was not right. We examine what went wrong, try to see what caused it, and start again.

In addition, it is also important for the teacher to understand that the teaching flow slightly differs according to the character of matters which are embedded in daily life or society.

The second approach is to present a real world problem for students and let them solve the problem. The intention is for the students themselves to construct the mathematical model. The teaching flow almost follows the full modelling process generally recognized as mentioned directly above. The teacher is expected to set a series of observations or actions by students themselves, then let students derive key questions which will become the intended problem presented by the teacher.

When a teacher presents the modelling task couched in verbal terms for students, we need to pay attention to the fact that the formulated problems couched in verbal terms are also regarded as matters which someone has developed in the past, while we should exclude word problems which are “dressing up” of purely mathematical problems in words referring to a segment of the real world. Therefore, the teacher should take enough time in teaching modelling so that students can interpret why and how the problem situation was generated, before tackling how to solve the problem. The problem is derived by others, not by students themselves. Interpreting formulated problems couched in verbal terms can be considered one of the teaching aims, in addition to meaningful attempts to solve the problem.

It is the teacher’s role to consider which approach is appropriate for students, by taking account of both the educational goals and students’ surroundings.

5. Summary

This paper discussed that concepts about “time” and “place” have an important role in the practical teaching and learning of applications and modelling, for the following three aspects: (1) Problem situations which people are interested in, (2) Purposes for using mathematics in a society, (3) Existence of mathematical models, methods, etc. embedded in a society.

Regarding (1), it is suggested for the teacher to understand that problem situations for people in a society and problem situations for students are different. Therefore, the following two points should be especially stressed in teaching. First is to examine the relation between the problem situation and students from perspectives such as “at present or in the future”, “for all students or for a few students” and “for citizens, for individuals, or for profession/vocation”. Second is how to introduce modelling tasks for students. It is suggested for the teacher to clarify the reason why students must solve the problem, or to set a series of observations or actions so that students can accept the problem posed by someone as their own problem.

Regarding (2), it is noted that the purposes for using mathematics in other disciplines or areas of practice are important for the following three points: (a) Educational goals students are expected to attain, (b) Understanding of modelling process for the beginner, (c) Appreciation of the usefulness of mathematics in society. For example, in (c), it is possible for the teacher to assess how students can deepen their appreciation about the usefulness of mathematics by setting the criteria based on the purposes to use mathematics: *understand*, *action* and *design*.

Regarding (3), we need to pay attention to the existence of a variety of matters familiar to students in daily life which were developed using mathematics. This fact leads to two quite different approaches to teaching mathematical modelling. The first approach is to present a matter which someone else has developed in the past to students and let them interpret it. The second is to present a real world problem for students and let them solve the problem. It is the teacher’s role to consider which approach is appropriate for students by taking into account both the educational goals and students’ surroundings.

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