Mathematical Knowledge for Secondary School Mathematics Teaching

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Recent research literature that has focused attention on mathematical knowledge for teaching posits that mathematics for teaching is a different kind of mathematical knowledge—deriving from classroom practice and different in nature from the mathematical knowledge commonly acquired in collegiate mathematics classrooms. This research (Ball, 2003; Ball & Bass, 2000; Ball, Bass, & Hill, 2004; Ball, Bass, Sleep, & Thames, 2005; Ball & Sleep, 2007), although intended to describe more generally the mathematics of teaching, has focused almost entirely on mathematical knowledge for teaching at the elementary level. The yet-to-be-addressed question was what the nature of mathematical knowledge for teaching at the secondary level would look like.

Although policy documents such as the CBMS report *The Mathematical Education of Teachers*(2001) and the MAA's *A Call for Change* (1991) indicate that the mathematics needed by secondary teachers should be deep and related to the mathematics they teach, little has been done to characterize that needed understanding. *A Call for Change* recommends the equivalent for secondary teachers of a collegiate major in mathematics, but one that differs considerably from the traditional program of courses in its focus and coverage. Work over the past few years at The Pennsylvania State University has focused on characterizing the mathematical knowledge and understandings of and for prospective secondary teachers[1]. The Penn State work has adopted three different lenses on knowledge of mathematics for teaching, each lens in a different study or series of studies: (1) a lens that focuses on mathematical knowledge as the mathematical thinking central to conceptually based curricula typical of reform in U.S. curricula (Heid, Perkinson, Peters, & Fratto, 2005; Heid, Lunt, Portnoy, & Zembat, 2006; Heid & Zembat, 2008; Portnoy, Heid, Lunt, & Zembat, 2005; Sullivan, Heid, & Akar, 2005) (2) a lens based in classroom practice that focuses on mathematical

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knowledge that teachers could use in the everyday work of teaching (Situations Group of CPMT and MAC-MTL); and (3) a lens based in mathematical knowledge as knowledge of mathematical processes (Zbiek, 2008). This manuscript will provide brief descriptions of each of these approaches.

Mathematical Knowledge for Teaching Secondary Mathematics: A Curriculum-Based Mathematical Thinking Approach

Mathematics education faculty, mathematics education graduate students, and a mathematician developed three courses designed to enhance the depth of mathematical understanding by secondary teachers of concepts and processes related to functions, data analysis, and mathematical modeling. These courses were intended to afford opportunities to prospective teachers to deepen their understanding of mathematical ideas central to secondary mathematics, including the mathematics underpinning emerging reform curricula. Based on interview and observation data gathered in the context of these three courses, faculty and graduate students at Penn State conducted a series of four studies (each based on a semester-long series of interviews and observations) to characterize the mathematical knowledge and understandings of prospective secondary mathematics teachers. These studies targeted understanding that went beyond common mathematical conceptions or misconceptions; they sought to describe the ways that prospective teachers think about mathematics in the context of secondary mathematics.

Understanding of entities as objects or inscriptions. In Study 1 Penn State researchers (Heid & Zembat, in progress) have studied the ways in which prospective secondary mathematics teachers' use of representations reveal their understanding of mathematical objects. Using prospective secondary mathematics teachers' work with non-standard representations of function as grounding (three rounds of interviews were conducted with each of 11 prospective secondary mathematics teachers), this study developed a schematic diagram that accounts for the range of connections that students make between mathematical concepts and the inscriptions commonly used to represent mathematical concepts and relationships. This study developed empirically supported characterization of prospective secondary mathematics teachers' understanding of mathematical entities by describing relationships between understanding of a mathematical object and use of external representations (inscriptions)– in the particular context of the mathematical concept of function. The study investigated how students understand function as revealed through their use of external representations in unfamiliar settings and through unfamiliar inscriptions. It focused on the relationships between their use of inscriptions and what the researchers took to be their understanding of the mathematical object (usually function) ostensibly represented. They developed ways to characterize understanding a mathematical entity as a configuration of mappings (Inscription-Object diagrams) between external representations and the idealized understanding of the mathematical object. These characterizations accentuate the importance of maintaining the meaning of subobjects in reasoning about objects (e.g., maintaining the notion of function as a mapping in recognizing function in three-dimensional settings), the importance of maintaining an object-inscription connection in moving among representational settings (e.g., keeping track of the identity of the input variable in dealing with function), and the importance of coordination of inscriptions and objects (e.g., simultaneously extracting information about a mathematical entity from different inscriptions and coordinating that information).

Understanding of complexity. Study 2 (Heid, Lunt, Portnoy, & Zembat, 2006) investigated the ways that prospective secondary mathematics teachers deal with problems that involve a number of variables and the relationships between them. Data consisted of annotated transcripts of task-based interviews conducted with each of 11 prospective secondary mathematics teachers (the same database that was used in Study 1). In the case of problems that involve a number of variables and intricate relationships between them, monitoring is needed to maintain simultaneous attention to the multitude of variables and their features while coordinating that attention. Solving this type of problem also requires monitoring when to focus on the global and when to focus on the local. Study 2 revealed productive and unproductive ways in which students worked with templates and prototypes, as well as their differing fluency in moving between a macro-perspective on the problem (one that kept the global relationships at the fore) and a micro-perspective (one that helped them successfully analyze component relationships in the problem).

Differing approaches to understanding mathematical concepts: The case of weighted mean. Study 3 (Sullivan, Heid, & Akar, 2005) investigated prospective teachers' understanding of the key statistical concept of weighted mean as revealed in the context of physical, contextual, numerical, algebraic, and graphical representations. Individual interviews were conducted with 15 prospective teachers, and small- and whole-group class activity sessions were recorded throughout the semester. Analysis of those data suggested that of particular importance to students' understanding of weighted mean was the nature of their governing perspective (e.g., procedural vs. conceptual or structural). Key to revealing their understandings were interview questions that required them to extend their processes to less familiar settings. Prospective teachers with a conceptual or structural approach were better able to extend their understanding to these unfamiliar settings (e.g., tasks that asked for the weighted mean of data expressed in two variables instead of one and tasks that asked students to extend the notion of mean as balance point to histograms).

Understanding intricately related concepts: The case of sampling distribution. Understanding of data analysis and statistics has in recent times taken on increasing importance in secondary mathematics, and core to understanding inferential statistics is the concept of sampling distribution. Study 4 (Heid, Perkinson, Peters, & Fratto, 2005) investigated the ways that prospective secondary mathematics teachers understand sampling distribution and its related concepts (e.g., distributions of sample means, population distributions, effects of changing sample sizes). Two rounds of task-based interviews were conducted with 18 prospective teachers. Small- and whole-group class activity sessions were recorded throughout the semester. Through its investigation of the ways that prospective teachers understand sampling distribution, this study developed characterizations that can generalize to the ways that prospective teachers make and manage distinctions among intricately interrelated mathematical concepts and relationships.

The focus of these studies was not so much on content of school mathematics but on ways of mathematical understanding and thinking that would address the needs of teaching secondary mathematics. Moreover, the ways of mathematical thinking documented in these studies have potential generalizability across secondary mathematics. Secondary mathematics introduces the need for teachers to develop and use a broad range of new external representational systems, and Study 1 suggests a ways that prospective teachers think about and use these external representations. Problem situations in secondary mathematics increasingly require accounting for simultaneous quantitative constraints, and Study 2 describes ways that prospective secondary mathematics teachers deal with this complexity. Secondary mathematics requires flexibility in dealing with an increasing range of manifestations of a mathematical concept, and Study 3 describes the power of a conceptual or structural approach in that setting. And, finally, secondary mathematics requires prospective teachers to make and use increasingly finer distinctions among related mathematical ideas, and Study 4 gives some insight into the ways prospective secondary mathematics teachers make and manage mathematical distinctions. These characterizations of the demands of secondary mathematics suggest just some of the mathematical needs of increasing importance at the secondary level. Studies such as those described in this section suggest a need for new ways of thinking about the mathematical knowledge needed by teachers at this level.

Mathematical Knowledge for Teaching Secondary Mathematics based in Classroom Practice: A Situations Approach

Recent research literature that has focused attention on mathematical knowledge for teaching posits that mathematics for teaching is a different kind of mathematical knowledge—deriving from classroom practice and different in nature from the mathematical knowledge commonly acquired in collegiate mathematics classrooms. This research, although intended to describe more generally the mathematics of teaching, has focused almost entirely on mathematical knowledge for teaching at the elementary level. The yet-to-be-addressed question was what the nature of mathematical knowledge for teaching at the secondary level would look like.

Over the past three years, Penn State faculty[2] and graduate students from the Mid-Atlantic Center have collaborated with University of Georgia faculty[3] and graduate

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[2] Penn State faculty engaged in this effort included M. Kathleen Heid, Glen Blume, and Rose Mary Zbiek.

[3] University of Georgia faculty engaged in this effort included Patricia Wilson, Jim Wilson, and Jeremy Kilpatrick.

students from the Center for Proficiency in Teaching Mathematics in the Mathematical Knowledge for Teaching at the Secondary Level Project (MKTS). The MKTS Project has as a goal the development of a practice-based framework for the mathematical knowledge that is useful in teaching mathematics at the secondary level (Situations Project , 2008). Through regular teleconferences, occasional face-to-face meetings, and work on joint presentations, the project has developed and vetted over 50 "Situations" that describe "mathematical opportunities," actual events occurring in the practice of teaching, and the mathematical knowledge that might be useful to teachers while taking action in the course of those events. The researchers convened a conference of experts, mathematicians and mathematics educators whose nationally and internationally known work has focused on the development of the mathematical knowledge of secondary mathematics teachers, to provide guidance on the development of a framework. Based on the advice offered during the conference, the researchers expanded the approach by developing descriptions of the "mathematical actions of teaching," actions taken by teachers that required the application of mathematical knowledge.

The framework for the mathematical knowledge that is useful in teaching mathematics at the secondary level exists only as a first draft and is still under construction. We have analyzed both the mathematical understandings that we identified in our Situations and the mathematical actions of teaching that were derived from several cross-university conferences. The current version of the framework develops three threads of Mathematical Proficiency for Teaching Secondary Mathematics: proficiency with content, proficiency in mathematical activity, and proficiency with teaching mathematics. These threads account for the mathematical knowledge and abilities teachers need to have in their repertoires, the particular ways of mathematical thinking that teachers can productively bring into play as they teach mathematics, and the mathematical knowledge that secondary teachers can productively draw on as they plan, implement, and reflect on mathematics lessons. One goal in the production of this framework is to provide guidance to those who teach mathematics on the ways of mathematical thinking that ought to be central to the mathematical experiences of secondary mathematics teachers.

The teaching of secondary mathematics presents mathematically prepared teachers with daily opportunities to use their mathematical knowledge to extend and enrich the mathematical experiences of their students. The MKTS project has identified types of mathematics that secondary teachers can productively draw on in the course of their mathematics instruction, and the project is in the process of developing a framework to characterize the mathematical knowledge that can be useful to secondary mathematics teachers. This framework can inform those who are engaged in the mathematical preparation of secondary teachers, providing benchmarks and suggesting overarching goals in providing teachers with background they can draw on in the work of teaching secondary mathematics.

Mathematical Knowledge for Teaching Secondary Mathematics based in Classroom Practice: A Mathematical Process Approach

A third perspective, and the one most central to the work of the Mid-Atlantic Center for Mathematics Teaching and Learning used by the Penn State research group to focus on mathematical knowledge for teaching secondary mathematics is mathematical knowledge for teaching as knowledge and use of mathematical processes. A multiyear study (during two consecutive semesters of methods classes, one semester of students teaching, and the first year of teaching) is aimed at characterizing knowledge of mathematical processes of prospective secondary mathematics teachers and how they use that knowledge in their teaching. The prospective teachers are participating in task-based interviews every semester of this 3-year project, and their teaching is observed during a pre-student teaching practicum, during student teaching, and during their first year of teaching. Task-based interviews that provide a look at prospective secondary mathematics teachers' uses of mathematical processes, and observations provide evidence about their use of processes in their teaching.

We chose to focus on process (e.g., Defining, Justifying) instead of on particular mathematical content for several reasons. First, we wanted to follow the prospective teachers through their methods and practicum experiences, including student teaching, and possibly into their first year of teaching. We knew that there was little chance that, on the date of a particular classroom observation, the prospective teachers would be teaching any particular mathematical topics that we might choose to assess, but that it was likely that, regardless of the particular lesson topic, they would call on general mathematical processes. We surveyed the research literature on understanding of mathematical processes related to mathematical thinking by college students. This literature fell into the categories of symbolizing/representing, justifying, defining, generalizing, and abstracting. Because we believed that we would see little abstracting in our observations of classrooms or in our task-based interviews, we focused our investigation on the mathematical processes of symbolizing/representing, proving/justifying, defining, and generalizing. A research team led by a faculty member (Zbiek et al., 2008) developed and refined our literature-based definitions of the targeted processes, and the entire research team is using these definitions to identify evidence of mathematical processes both in the annotated transcripts of interviews and in the annotated transcripts of classroom observations.

We are currently in the process of developing case studies of individual teacher candidates that will account for the relationships between understanding of mathematical processes of prospective teachers and how they use these processes in their teaching. The central goals of this work will be to document mathematical understanding for teaching and to investigate ways that such understanding is or is not applied as beginning mathematics teachers confront the day-to-day challenges of teaching secondary school mathematics. Cross-case analyses will suggest ways to understand how prospective teachers develop in their ability to understand and implement mathematical processes that are useful in teaching.

The Mathematical Process approach to mathematical knowledge for teaching focuses on characterizing the ways of offers a way to think about mathematical knowledge for teaching secondary mathematics that transcends the particular collegiate mathematics courses that secondary teachers typically complete. Characterizations of the processes secondary teachers use in their teaching can help inform the mathematical activity needed in collegiate mathematics courses they complete as part of their preparation for teaching.

Conclusions

This paper has provided a brief overview of the approaches that researchers at The Pennsylvania State University have taken to understanding and characterizing mathematical knowledge for teaching secondary mathematics. Each of these approaches has an empirical base on which theory is being built. Rather than thinking of the approaches as disjoint choices, it seems most productive to consider all three approaches in the effort to define mathematical knowledge for teaching. We have used several different methodological approaches. The Curriculum-Based Mathematical Thinking approach is based on in-depth analysis of the work of prospective teachers in the context of mathematics classes and task-based interviews. The Situations approach is based on the documentation of actual secondary mathematics classroom situations and the mathematical analysis of those situations. The Process approach is based on task-based interviews, and cycles of classroom observations that include interviews and observations. As the field gains a better understanding of the mathematical knowledge that secondary mathematics teachers can productively use in their teaching, it will need to turn its attention to creating ways to help teachers develop that knowledge. That work is in its beginning stages.

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