

# Reflections on Teaching Elementary Arithmetic: Implications for Understanding Number Theory and Discrete Mathematics

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*All students deserve high-quality programs that include significant mathematics presented in a manner that respects both the mathematics and the nature of young children. These programs must build on and extend students' intuitive and informal mathematical knowledge. They must be grounded in a knowledge of child development and provide environments that encourage students to be active learners and accept new challenges. They need to develop a strong conceptual framework while encouraging and developing students' skills and their natural inclination to solve problems.*  
— NCTM Principles and Standards

## INTRODUCTION

The recommended focal standards for teaching mathematics in Pre-K through Grade 2, according to the NCTM's *Principles and Standards for School Mathematics* are “Number and Operations” and “Geometry.” As for the Content Standards for grades 3-5, the *NCTM Principles and Standards* identifies “three crucial mathematical themes—multiplicative thinking, equivalence, and computational fluency.”

The NCTM therein also recognizes, pedagogically, that “[i]t is essential for students in the elementary grades to study mathematics ... under the guidance of teachers who enjoy mathematics and are prepared to teach it well.” Overall, I take the general tenor of the *NCTM Principles and Standards* to be that teaching and learning whole numbers in the early grades are ultimately subservient to teaching and learning real numbers and related topics such as fractions, decimals, percentages, measurement, and applications using rational and real numbers.

In this paper, I wish to argue that teaching and learning whole number arithmetic and elementary number theory form a necessary component to teaching and learning real and rational numbers and related topics, but also, more specifically, discrete mathematics, and quite likely aspects of other areas of mathematics, such as combinatorics as well.

Preservice teachers are generally not at all knowledgeable about the conceptual, viz., number theoretical, aspects of whole numbers, their relations to rational and real number representations, viz., fractions and decimals. Moreover, they are typically quite insecure and anxious about their understandings of elementary arithmetic, if not when they begin their preservice education, then when it becomes evident to them that they have little to no understanding of the formal or conceptual connections between such constructs.

I have written about these matters in more scholarly manners elsewhere, and recommend the reader to pursue those readings in the references as they feel so inclined. My approach here is more anecdotal, although informed by my research and scholarship in this area, I will be drawing mainly upon my experience as a teacher educator.

## WHAT IS A NUMBER?

This is one of my favorite opening questions with any new cohort of preservice teachers. Students confidently pipe up with responses such as “42,” “7,” “pi,” “ $\frac{1}{2}$ ,” “.5,” and occasionally more general statements, such as “a number is a quantity.”

I typically like to follow up the more specific responses with a question like “what is the difference between X and Y?” where X is a whole number and Y is a rational or real or decimal number from the set provided by the students in response to my first question.

I ask this question concerning differences first because I am most likely to be met with blank stares, at least for a few moments. Some students may venture answers such as “42 is made of numbers and pi is made of letters,” “ $\frac{1}{2}$  is a fraction and 7 is a whole number,” and “the difference between 7 and  $\frac{1}{2}$  is 6 and  $\frac{1}{2}$ .”

Not wanting to neglect those students venturing forth with more general responses to my opening query as to what a number is, I will ask “what is common between all of these answers that you have provided? I am particularly gratified if someone offers the following obvious response: “They are all numbers.”

I can then, with my most Socratic air, come full circle: “These are all examples of numbers, but I ask you all again, what is a number?” If someone had ventured forth with a more general response, such as “a number is a quantity,” this is my moment to gaze straight at them, with an encouraging nod, if necessary, hoping to evoke that more general response again, and to follow up accordingly.

Providing definitions, discerning differences, and making connections, for and between arithmetic concepts, in manners that can be conceptually articulated, is not, in my experience, a strength for most preservice elementary school teachers.

As important as I think it is for students, in the constructivist tradition, to construct their own understandings, I view my role as a teacher educator to help students develop and reconcile their conceptual understandings of mathematical concepts with the mathematical formalisms they will be responsible for teaching.

First, and foremost to my mind, is to help students understand that a number is a concept. What, then, is a concept? Well, concepts result from abstracting and generalizing. I find many of my students cringe, however, when words like concepts and abstractions are thrown around, and typically for good reason. They are usually empty shells.

Task number one, then: concepts. Let’s talk first about what concepts are, how concepts are formed, and what makes a concept a number concept, and then it might make more sense to talk about number theory, that is, about how number concepts can be distinguished, related, and understood, if not rigorously formalized.

So I ask: “What is a concept?” I love it when I get answers like “number is a concept” or “a concept is an idea.” We go through a similar dialectic on another level. Some students are shocked when I assert that ‘chair’ is a concept, ‘table’ is a concept,” or even that ‘cup’ is a concept.

The follow up is: “What makes a concept a concept?” Again, I am usually met with blank stares with this question. How could thinking about things so familiar now seem so strange? What is so familiar, and yet what is so strange to this population, is *thinking*.

I admit to being irked with what has become a widely adopted teacher education mantra “tell me, and I forget; show me, and I remember; involve me, and I understand.” To me, this saying may also indicate that students are underdeveloped in the basic faculties of thinking: memory, imagination, and reasoning, respectively.

How to engage students in thinking? Asking the question “what makes a concept a concept?” has in my experience offered an excellent start. I don’t expect my students to be able to come up with answers to this question. It is rhetorical. It is posed to help them become conscious of the fact that, despite 16+ years of schooling, they have yet to become very knowledgeable of the nature of the basic atoms of thought.

First point: Thinking is quite different from perceiving. I can see or hear or touch or taste or feel something, or I can think about what I see, hear, smell, taste, or feel. Second point: I can see, hear, smell, taste, and feel *this* cup, and I can think about what makes a cup a cup. Is it the color? Is it the sound? Is it the smell? Is it the taste? Is it the feeling?

Again, these questions are mainly rhetorical. We can get into a nice conversation about what a cup *is*, that is, about how to define the word ‘cup’ as a concept. When we preclude or separate some characteristics of a specific object from our consideration in defining a concept, we are abstracting.

Through the course of abstracting characteristics from a specific cup in defining a concept like ‘cup’, we are generalizing. This is because as we preclude, say, the color or pattern on a particular cup from our concept of cup, then any other cups with different colors and patterns, all else being equal, can fall under that concept of cup.

Second point: Thinking about concepts like ‘cup’ is quite different from thinking about concepts like ‘number’. I can see, hear, smell, taste, or feel a cup, that is, an instance of a cup, but how can I see, hear, smell, taste, or feel a number? What is it about the concept of number that is different from the concepts of chair, table, or cup?

These questions are not so rhetorical. Actually, they speak to differences between what in Piagetian terms is often referred to as concrete versus reflective abstraction. In concrete abstraction, the particulars of a concept are objects of sense. Those concepts themselves are intellectual objects. Abstraction over intellectual objects is reflective abstraction.

If all of this sounds somewhat abstruse and recondite, I invite the reader to consider Plato’s emphasis on the interdependent importance of similarity and difference when it comes to the nature of the human soul, psyche, or intellect. Indeed, it is not untoward to consider similarity and difference, in comparing one thing and another, as foundational aspects of rationality that are certainly fall within the intellectual grasp of children, in manners that are intuitively accessible, in that they draw on lived experience.

What is more accessible for elementary school kids than comparing everyday objects to identify similarities and differences? What is similar between this cup and that cup? What is different between this cup and that cup? What is similar between this book and that blackboard...? Finally, what is similar between *all* possible such objects?

What really gets interesting, and relevant to our study of number theory, progressive abstraction and generalization through reflection on objects and their attributes, on their differences and similarities, is to think of the concept of an external perceptual object in general. The only characteristics external objects share, the forms of perception in Kantian terms, are extension in space and duration in time.

I leave it to the reader, as I do my students, to reflect for a few moments upon what kind of object one would have were those last two perceptual attributes or characteristics of external objects are removed (i.e., abstracted) from consideration.

The kind of general object that one would have would be an object that one could only instantiate in one's mind. Such an object would be a purely intellectual object. In fact, it would be an arithmetic unit. There is a sense in which the arithmetic unit, conceived of in this way, is now a pure abstraction, a pure object of thought. Its instantiations are external perceptual objects. Hence, the unit applies to the perceptual world in very general ways, and not surprisingly, arithmetic as well.

What is the point these first two points? In sum, the first point is that there is an important distinction to be made between objects of perception and objects of intellect. The second point is that there is an important distinction to be made between concrete objects of intellect, like cups, and pure objects of intellect, like the arithmetic unit.

So what then, is a number? According to Euclid, number is defined as a multitude of units. I raise this famous definition as a case in point to further my considerations below. It is hard to imagine much meaningful progress being made in the teaching and learning of arithmetic without some *conceptual* basis for number and units. Alas, this basis is seldom found in preservice teachers of elementary school mathematics.

Indeed, it is hard to imagine making much meaningful progress in the teaching and learning of discrete mathematics, which is often viewed or defined as “applied number theory,” without some conceptual basis for number and units. Unfortunately, even with such a conceptual basis as just articulated in place, many confusions and confluences abound for the unwary. I turn to discuss some of those now.

### **WHAT IS A WHOLE NUMBER?**

We return now to some answers to my initial question: 7 and 42, for instance, are whole numbers. Extensionally, whole numbers are elements of the set  $\{0, 1, 2, \dots\}$  (imagining, that is, that one could actually write out such an infinite list). Intensionally, a whole number is a number from which successive subtractions of 1 eventually yield 0.

More commonly considered, a whole number is an element of the set of whole numbers, which in turn is a subset of the integers, which in turn is a subset of the rational numbers,

which in turn is a subset of the real numbers, which in turn, is a subset of the complex numbers... This approach does not make the question “what is a number” easier for elementary school children, let alone elementary school teachers of mathematics.

Just sticking with whole numbers, we have some tricky questions to address. Neither Euclid nor his ancient and classical compatriots had any concept of 0 — for how, they pondered, could nothing exist? How could nothing possibly be a something? For that matter, it is helpful to note that according to Euclid’s definition, 1 is *not* a number.

Sticking with Euclid, we may be able to work our way out of these difficulties by defining the unit as a special number that we can call, well, a unit. A multitude of one may sound like a contradiction of terms, but it is one we can live with, much as we do with that famous trinity, ‘me, myself, and I’. For kids, I’m sure they would be thrilled thinking even further along these lines, about 0 as a multitude of none!

What is the alternative for elementary school children, or for preservice teachers of elementary school teachers, for that matter: Dedekind cuts, perhaps? I think not! A much more intuitive distinction when it comes to distinguishing between whole numbers and rational numbers (viz. fractional numbers), and one that is more readily accessible to kids, is that whole number units cannot be divided. The Greeks, by the way, were adamant about this. Rational number units, however, can be divided.

What is it, then, that best characterizes a whole number? I claim it is that, at least when it comes to elementary school and the ancient and classical origins of number theory, whole numbers consist of indivisible units. That is to say, the units that constitute whole numbers cannot be divided.

I can imagine raising the chagrin of many in such an august audience with such a claim. It would seem there is a sense in which I am advocating setting mathematics back by a couple of millennia. Well, truth be told, yes, with respect to elementary mathematics education, I am advocating something quite like that.

We have confused the heck out of elementary school kids and teachers alike by expecting them to grasp, at an elementary level, two millennia of advances in formal mathematics. Perhaps the chagrin can be mitigated reflecting on the absurdity of such an approach.

What is it that elementary kids and teachers find so confusing? Nothing less is more confusing for this population than understanding relations between whole numbers, fractions, decimals, and percentages.

So, let’s try to introduce some conceptual clarity by distinguishing between arithmetic units that *cannot* be divided, and other arithmetic units that *can* be divided (viz., in a very special way) and see how far along it gets us. In my experience, I have found, both conceptually and pedagogically, that it gets us a very long way.

Whole numbers, then, are made of units that cannot be divided. That is why we have division with remainder. As a matter of fact, just because of that form of division, we can talk about whole number arithmetic. Indeed, I claim we can go further. We can then talk

about number theory and discrete mathematics. I now provide a few choice examples.

Consider multiplication tables. One of the great revelations in my experience as a teacher educator was just how few preservice teachers realized that a multiplication table is just a truncated table of multiples. What?! It is also, now, one of the great joys of my teaching, helping preservice teachers make conceptual connections where none existed before.

Some of the questions I have used in my research that have helped lead me to such revelations are questions like: Consider the five digit numbers 10651 and 10661: Is there a number between these two numbers that is divisible by 7? The range of responses my colleagues and I have received from preservice teachers to this kind of question is remarkable. Some actually divide every number before answering.

How could someone with 16+ years of schooling *not* answer “yes, there is at least one, and perhaps two?” An answer to *this* question is readily apparent if one is lacking in the most rudimentary understanding of basic concepts from elementary number theory. The division theorem, commonly referred to by mathematicians as the ‘division algorithm’, states that for every whole number dividend  $A$  and non-zero divisor  $D$ , there exists a unique quotient  $Q$  and remainder  $R$  where  $0 \leq R < D$ , such that  $A = QD + R$ .

Whole number division, viz., division with remainder, rests on repeated applications of the division theorem. Divisibility, pertains to those cases of the division theorem where  $R = 0$ . Here, connections between multiples, divisors, and factors, become evident.

Indeed, the division theorem provides much of the basis for the multiplicative and additive structure of whole numbers. I am aghast to read of mathematicians and mathematics educators advocating we dispense with long division.

Having elementary school kids work with whole numbers, and exploring the unending plethora of relations and structures to be found amongst and within them, is a royal road to engendering endless fascination with numbers and deep understandings of them. To do any less, to confuse kids with unconnected and conflated concepts serves the opposite effect. Imagine, rewarding kids for doing what they are told without understanding. Mental enslavement, I say, not intellectual enlightenment.

#### **WHAT IS A RATIONAL NUMBER?**

When we talk about numbers like fractions and decimals, numbers that are made of units that can be divided, divided, that is, in a very special way, (viz., into equal parts), then we have a different kind of division (though similar in some ways), and we have a different kind of arithmetic that, for elementary school, I call rational number arithmetic.

Is it such a difficult discernment for elementary school teachers and learners alike to make? Whereas whole numbers consist of units that cannot be divided, rational numbers like fractions and decimals consist of units that can be divided. Any child recognizes that grandma’s antique lamp loses, shall we say, some “integrity,” that is, shall we say, its “integral” nature, if it is broken into pieces?

Any child that has argued that their sibling's half of the cookie is bigger than theirs, knows that the fairest way to divide the cookie is for the two halves to be exactly equal. Alas, it is impossible, when it comes to external objects, to divide any of them *exactly*.

In our minds, though, when it comes to an arithmetic unit, we can imagine that such a form of division is possible. That form of division that involves dividing an arithmetic unit into a whole number of parts that are, each and every one, exactly equal, I call 'denomination'. Then we can count, or enumerate, those resultant parts.

Where, then, is the mystery in referring to the denominators and numerators? Denominating the unit and numerating the resulting parts forms the whole number basis of the mathematical concept of rational numbers we refer to as fractions. Indeed, ratios are the comparison of one whole number in terms of another whole number.

### **WHAT IS A REAL NUMBER?**

A further interesting consideration is to question how all this may be extended and relevant to real number units. Formally, of course, so far as rational numbers and whole numbers are defined as a subset of real numbers, such questions are simply not relevant. From psychological and pedagogical perspectives, as we have seen, such questions are relevant. So it is appropriate for us to continue along these lines and ask: are rational number units the same as real number units? Or, in what ways may they differ?

Just as we saw that whole and rational number units can clearly be distinguished in a conceptual sense by the property of denomination, viz., the arithmetic units comprising rational numbers can in principle be divided into equal parts and units comprising whole numbers cannot, is there a similar or related property that we can use to distinguish units comprising rational numbers from units comprising real numbers?

Clearly, real numbers that are also rational numbers can be denominated. For such cases, the property of denomination applies to both, and so, denomination as already considered may not be sufficient to conceptually distinguish between rational and real number units. It is interesting to note that not denominating the arithmetic unit, i.e., by leaving it whole, can be taken as a degenerate case of finite denomination, which we might choose to call *unity*. In this light, we can think of the denomination of rational numbers, at least in the sense of fractions, in terms of finite denomination, and extend the concept such that whole number denomination is restricted to unity.

This refined notion of denomination, including the degenerate case of unity, becomes consistent with the formal notion that whole numbers comprise a subset of the rational numbers. Continuing along this vein, the notion of denomination can be extended to include infinite denomination. This extended concept of denomination allows us to characterize real numbers in terms of units that can be infinitely denominated.

What does it mean, though, to divide a unit into an infinite number of parts? Such considerations are far from intuitive and are left, for now, to further reflection.

### **CONCLUDING REMARKS**

When there is a focus on teaching and learning the conceptual underpinnings and structure of whole number arithmetic, grounded on elementary school children's lived experience, their intuitive and informal mathematical knowledge, combined with knowledge of concept formation and child development, much improvement in understanding basic number theory and discrete mathematics may result.

When we allow denomination, we allow a fundamentally new form of division. It is important, or so it seems to me, that we help elementary school kids to understand that, and that we begin by helping preservice elementary school teachers of mathematics to understand that they are responsible for teaching two related, but conceptually quite different arithmetics. One, whole number arithmetic, concerns basic concepts of elementary number theory. Whole number arithmetic is the more intuitive and intellectually accessible arithmetic, and the basis for discrete mathematics.

## REFERENCES

- Campbell, S. R. (2006). Understanding elementary number theory in relation to arithmetic and algebra. In R. Zazkis & S. R. Campbell (Eds.) *Number theory and mathematics education: Perspectives and prospects*. Lawrence Erlbaum Associates.
- Campbell, S. R. (2004). Theory and practice in Plato's psychology of mathematics education. *Philosophy of Mathematics Education Journal*, 18.
- Campbell, S. R. (2003). Dynamic tracking of preservice teachers' experiences with computer-based mathematics learning environments. *Mathematics Education Research Journal*, 15(1), 70-82.
- Campbell, S. R. (2002). Early mathematics education: A case of the blind leading the blind? *Philosophy of Mathematics Education Journal*, 15.
- Campbell, S. R. (2002). Coming to terms with division: Preservice teachers' understanding. In S. R. Campbell and R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction*, pp. 15-40. Westport, CT: Ablex.
- National Council for Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA, U.S.A.: Author.