Learners' conceptions in different class situations around Königsberg's bridges problem

L. Cartier – Institut Fourier (Grenoble, France) and « maths à modeler » research group
J. Moncel – Laboratoire G-SCOP (Grenoble, France) and « maths à modeler » research group

Abstract: This paper focuses on the so-called Königsberg's bridges problem. This famous problem is often used as an introduction for graph theory and discrete mathematics. It is also frequently proposed as an "enigma" in "recreational mathematics". However, its complete solution is rarely given and its mathematical depth is usually masked. Indeed, although the problem is nowadays completely solved, it remains a subtle and interesting one, giving access to fundamental mathematical concepts like proof, necessary and sufficient conditions, modelling. We propose several experiments. Three of them we made using a "research problem" approach with pupils in primary school, students in secondary school and math teacher candidates; and another one was a document study with undergraduate students in computer science and applied mathematics. We show that, whereas the problem and its solution being (apparently) simple, pupils and students at various levels encounter the same difficulties on some specific points that we detail, concerning proving and modelling. In addition, a careful look at Leonhard Euler's proof reveals that he might have encountered the same difficulties, an important part of the proof being actually missing.

Keywords: didactics of mathematics, discrete mathematics, graph theory, Königsberg's bridges problem, Euler's theorem, Eulerian path, proof, modelling, research problem, document study.

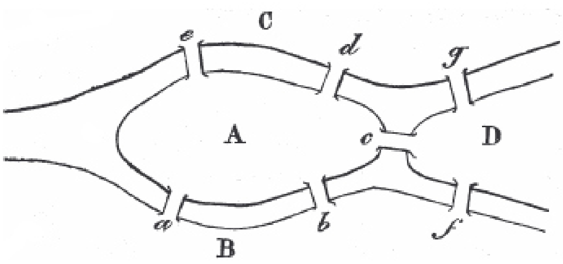
Introduction

In 1736 Euler published a paper entitled *Solutio problematis ad geometriam situs pertinentis* (the solution of a problem relating to the geometry of position). This paper became in some sense "legendary" as the seminal paper of graph theory and algebraical topology. Having a close look at it is interesting for us as teachers. Indeed we can study through this paper the way maths were written at this time, the model, the method and the arguments of Euler, the emergence of a model and its non-uniqueness and even, the difficulties with necessary and sufficient conditions. As we shall see, it may also be used with students, e.g. for document study activities. The focus of Euler's paper is the Königsberg's bridges problem, which belongs to the class of equivalent problems solved by Eulerian paths in graphs.
After giving a brief history of the problem, we will describe shortly several experiments in classes based on this type of problems and then we will focus on the learner's *conceptions* (see VERGNAUD 1991). The framework of this study is based on the methodology and the spirit of French didactics in mathematics (for a glossary and an a presentation of the domain in English, see papers in LABORDE 2005). This work is motivated by the fact that graph theory (including Eulerian graphs) became a part of the French secondary school curriculum in 2002. This raised new challenges for the teachers, in particular because most of them had to teach a topic they never learned themselves. This paper is part of a research project aiming at identifying difficulties (both for the teachers and for the learners) and developing new tools for the teaching of graph theory in the French secondary school. Our approach is that of the so-called *"research situation"* (*"situation recherche"*) described by Grenier and Payan (GRENIER 1998), currently used within the "maths à modeler" research team.

A brief history of the problem

In 1735, Ehler posed the Königsberg's bridges problem to Euler, as the possible first example of geometry of position[1] (i.e. “not involving measurements nor calculations made with them”). He then published a paper (EULER 1736) solving it. This paper is often considered as the first one in graph theory (see for instance BONDY 1976). Euler is even sometimes considered as having solved the problem using graphs (see DASGUPTA 2006).

Euler explained the problem in that way (English translation from BIGGS 1976[2]):



The problem, which I am told is widely know, is as follows; in Königsberg in Prussia there is an island *A*, called the Kneiphof; the river which surrounds it is divided into two branches, [...] and these branches are crossed by seven bridges *a*, *b*, *c*, *d*, *e*, *f* and *g*. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once.

Euler showed that there was no such route in Königsberg, using the fact that the numbers of bridges coming in each bank are all odd. He also claimed:

> If there are no more than two areas to which an odd number of bridges lead, then such a journey is impossible.
> If, however, the number of bridges is odd for exactly two areas, then a journey is possible if it starts in either of these areas.
> If, finally, there no areas to which an odd number of bridges leads, then the required journey can be accomplished starting from any area.

At the present time, these three propositions correspond to what is known as “Euler's theorem” in graph theory[3]. Euler proved that the condition “there are 0 or 2 areas to which and odd number of bridges lead” is a necessary condition (NC) for the existence of a path crossing each bridge once and exactly once, but he did not prove that it is a sufficient condition (SC) (even if he claimed it).
He used a model with sequences of letters corresponding to banks. His arguments consisted in computing the number of occurrences of the letters, as well as the number of bridges leading to a bank. Then he studies parity conditions on these numbers.

Later, C. Hierholzer, apparently unaware of Euler's paper, gave the first complete proof of an equivalent problem, “on the possibility of traversing a line-system without repetition or discontinuity” (see HIERHOLZER 1873). Finally, the first graph modelling the Königsberg's bridges problem appears in a so-called “mathematical recreation” (see ROUSE BALL 1892) .

[1]See (SACHS 1988).
[2]French translations in COUPY (1851) or in LUCAS (1882).
[3]The connectivity is not mentioned in this first theorem, but the question of crossing each bridge of the city, implies that a trail between two areas is always possible, that is to say that connectivity is implicit.

Experiments

“Research problem” situations

The general framework of what we call “research problem” situation is the one when a problem is given to the learners, who are asked to address and try to solve it, in the sense that they are expected to find related conjectures and arguments supporting them. The *didactic contract* is generally that, whereas answers may be partial (e.g., the learners may solve the problem only in some special cases), arguments have to be as complete and accurate as possible. Three series of experiments were carried, using three “research problem” situations.

A material game with strings and sticks

In this experiment we used a material game, consisting of a wooden board on which are fixed small sticks (see Figure 1). Black lines are drawn on the board. The problem is to find a path that goes from stick to stick without repetition, such that the “trace” of the string is exactly the picture drawn with the black lines. The path can start from any stick.

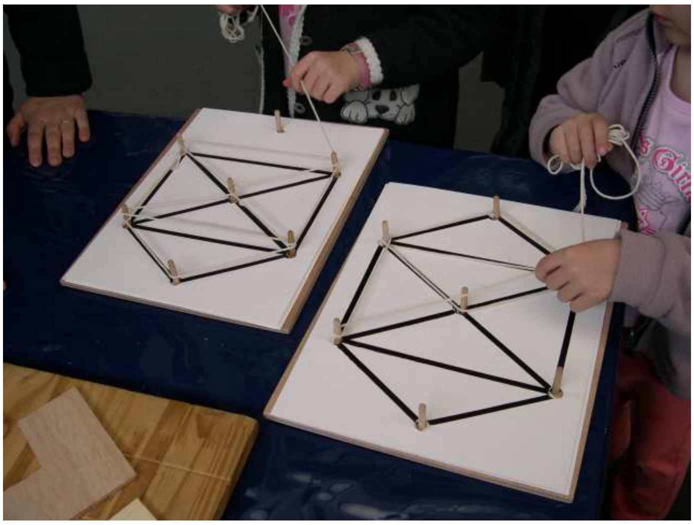


Figure 1: Pupils with strings and sticks games

Several boards are available, some for which such a path exists, some for which no such path exists. If such a path exists, the learners are then asked if there exists such a path which goes back to its starting stick. Here again, for some boards the answer is yes, for others the answer is no. This game was used in public scientific activities, like the “fête de la science” (the national week for sciences in France). The learners were mostly pupils and secondary school students, coming in class, and other persons coming individually. Around one hundred persons were observed trying out the game in this framework. This game was also used for a one hour group activity with 15 pupils of primary school aged between 5 and 8.

A museum tour and a graph problem

In the museum problem, several maps of museums are given to the learners (see Figure 2), who are asked to find a path that visits each room once and exactly once. The rule is that the path can go from room A to room B if and only if there is a door between rooms A and B. In the graph problem, several pictures of graphs are given to the learners, who are asked to find a path that uses each edge once and exactly once. The experiment consists in both problems. During week 1, the 20 secondary school students (aged between 15 and 16) worked on the museum problem.

In week 2, they worked on the graph problem with “big” (i.e. of around 20 vertices) and “tough” graphs. By “tough” we mean graphs having a “hidden” bridge (that is, an edge whose removal disconnects the graph), or non-connected graphs, but in a tentative masked way (see Figure 3). Each session lasted 90 minutes.

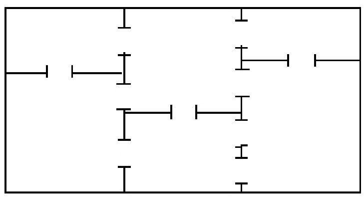


Figure 2: A museum

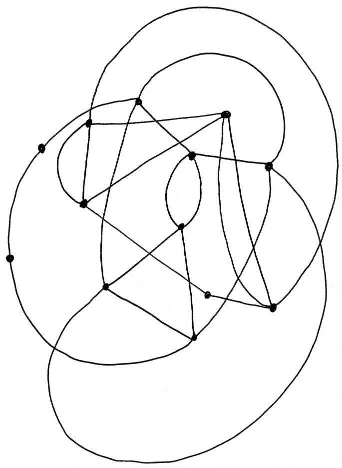


Figure 3: A tough graph

The Kaliningrad’s bridges problem

This problem is the Königsberg’s bridges problem, where the city has been renamed to Kaliningrad (the actual name of the city since 1946). It was given to 3 classes of 25 secondary school math teacher candidates, who were to know the problem, which belong to the official French curriculum. Each teacher worked about 15-20 minutes on the problem.

Document study and debate

This experiment was made with a class of fifteen undergraduate students preparing a master’s degree in computer science and applied mathematics, having a solid background in mathematics. In addition, they studied Euler’s theorem and its proof in two different courses prior to the experimentation. Thus, these students can be considered as experienced in discrete mathematics and in this specific problem.
As a personal homework, these students were asked to read and comment Euler’s paper (EULER 1736), in the French translation of E. Coupy (COUPY 1851). The subject was the following:
“*This paper is often considered as the first one in graph theory and algebraic topology. Read it, focusing on the reasoning, the modelling and the argumentation. What do you think about it? Are there some points that make you react? Bring out one or more problematics and develop them.*”
The students had one month to do the homework. A one hour debate took place two weeks after the homework due date.

Problematics

We will show that some questioning of the learners that appear during the “research problem” situations appear explicitly in the document studies. Moreover, several difficulties and arguments emerge with learners of all the ages, and with different mathematical backgrounds. They also seem to emerge independently of the way the problem was posed. In that sense, we claim that these experiments reveal the existence of *transversal* conceptions concerning argumentation and proof. In the sequel we focus on the following conceptions:

1) what is the difference between necessary and sufficient conditions?
2) what is a proof?
3) what is a good model?

We note that these results corroborate the arguments given in (GRENIER 1998) and in (ROLLAND 1999), about the specificity of discrete mathematics, and its relevance for the *devolution* of learners and for the learning of proof and modelling.

Learner's conceptions that appear during the experiments

How to distinguish necessary and sufficient conditions?

An important feature of the "research problem" situations described above is the fact that the learners quickly conjecture that the parity and the connectivity conditions are necessary and sufficient ones. Usually, solid arguments for the conditions being *necessary* are given. But there are generally considerably fewer and less convincing arguments for the conditions being *sufficient*. The first and most obvious interpretation is that there is an intrinsic difficulty to distinguish between a necessary condition (NC) and a sufficient condition (SC). Indeed, even the experienced students participating to the document study and debate experiment had difficulties with this point. Half of them did not notice that Euler did not prove the SC. After the debate, four of them (over 15) were still convinced that Euler's proof was complete. The debate revealed that the distinction between NC and SC was that not clear for the majority. In addition, we could even argue that, in some sense, Euler himself was not very clear on that point (see for instance §10-11 and §21 in EULER 1736, COUPY 1851, or BIGGS 1976). The difficulty of distinguishing NC's from SC's may come form the natural language itself, that uses sometimes the logic connectors in a rather confusing way (for instance, the threat "If you don't clean up your room, you won't get any chocolate" is usually interpreted as the promise "Clean up your room, and then you'll get some chocolate" - which is, logically speaking, false).

For this specific problem, however, we claim that there are other parameters that makes it difficult for the learners to find convincing arguments for the conjectured conditions being sufficient. In particular, we think that there is not enough motivation for the learners to prove anything, since a path is usually easily found, even with the "tough" graphs depicted in Figure 2. In that sense, we define the notion of *truth stake* ("*enjeu de vérité*"), that measures one's motivations for finding arguments to prove a given claim. In this situation, the truth stake for the necessity of *proving something* seem not easy to be stimulated, a path being always "easily" found by the learners if the conditions are satisfied. Some even conjectured that such a path could be greedily constructed (in the algorithmic sense).

What is a proof?

Experiments with primary school pupils and secondary school students reveal that there is a kind of *continuous process* from finding a solution, to constructing a formal proof. This process usually goes through the following steps: find a solution → emit a conjecture → convince oneself it is true → convince a classmate it is true → convince the teacher it is true → produce a written argumentation. The threshold after which the conjecture is considered as "true" usually varies between convincing oneself it is true and convincing the teacher it is true. In that sense, their notion of proof is close to the one defined by Wittgenstein: "What convinces us - *that* is the proof: a configuration that does not convince us is not the proof, even when it can be shewn to exemplify the proved proposition" (WITTGENSTEIN 1978).

As for the students who did the document study and debate experiment, they defined a proof as "a sequence of small steps between claims, that everyone can recognize as true". The last claim is the statement of the theorem, and the first one is the hypothesis ; this framework corresponds to the so-called *Euclidean deductive reasoning*. The students then realized that this "definition" was rather vague, in particular because one has then to decide if the steps are "small" enough to be considered as a proof. They concluded that the notion of proof could not be universal: something considered as a proof by person A might not be considered as a proof by person B. Hence, in some sense, these students also allow some *continuous variations* of the threshold after which an argumentation is considered as a proof.

Another phenomenon that we observed during these experiments is the fact that the activity of proof-making was generally considered as not being part of the mathematical activity. All the pupils and the secondary school students even considered that the "research problems" we proposed them did not belong to mathematics (even though the experiments occurred during a class of mathematics!). Usually, they considered that "it's not maths, because it's not geometry and there are no calculations". When we insisted on the fact that they were giving arguments that might look like proofs they studied in mathematics, they generally replied that "that's still not maths, that's logics". In addition, two undergraduates defended the idea, in their document study, that "the argument itself is not of interest, only the result is important", and that "mathematics have to give answers". Considering the mathematical background that theses students had to have, this is an additional argument to claim that the question "what is a proof?" is a relevant one. Indeed, even having a solid mathematical background does not make this problematics vanish, and similar conceptions are shared by learners of all ages and stages.

What is a good model?

For most of the "research problem" situations we experimented, there was no real necessity to construct a model to solve the problem. Indeed, conjectures and arguments for the museum situation or for the strings and sticks game can be found without modelling the problem with a graph. In the document study, all the students commented the quality and the relevance of the model proposed by Euler in his article. All the students considered that his model was "good, clever and relevant"[4]. Interestingly, half of them claimed that this could be claimed *a priori*, whereas the other half argued that the quality of the model could only be evaluated after Euler explained how to solve the problem with his model. Some were surprised by the fact that Euler "seems to focus on the banks, whereas the problem is about bridges", but they did not investigate further the question of what happens if we try to construct a model focusing on the bridges. In the experiment with maths teacher candidates, where the subject was precisely to construct a model (using a graph), half of them constructed a model focusing on the bridges. In this model, a bridge corresponds to a vertex, and the learners reduce the problem to finding a path that go through each vertex once and exactly once (such a path is usually called a *Hamiltonian path*). As one could expect, none of the learners who constructed this "Hamiltonian model" was able to solve the problem (see insert for further explanations).

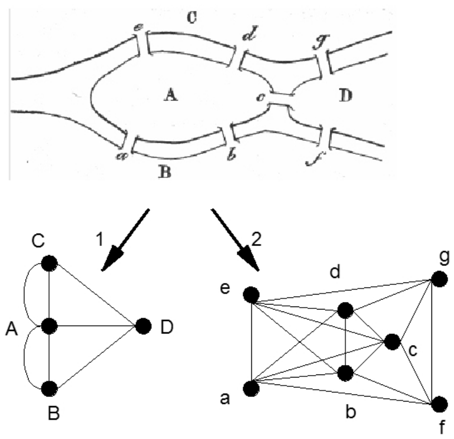
In the French manuals for secondary school students, the construction of a graph model is a topic that is often treated in a biased way, in the sense that the model is usually given to the learners, who have to notice that the proposed model is indeed a good one (see CARTIER 2003). Instead of model-making, the learners rather do a "model validity-checking" activity. Our experiments demonstrate that "non-biased" model-making (i.e. the learners construct themselves a graph and translate the initial problem into a graph problem, without being influenced by the teacher or the way the problem is posed) is a fruitful activity for learners, who experience something similar to professional mathematicians when they carry out their research, that is to say that modelling is an indispensable but non-trivial activity. The relevance of this specific problem (and of discrete mathematics in general) for mathematical education, is that it enables learners of all ages and of all mathematical backgrounds to quickly enter into such a research activity.

---

[4] It is to be noticed that half our students considered that Euler "solved the problem using a graph".

Hamilton or Euler?
Modelling the Königsberg's bridges problem with a graph can be done in at least two different ways:

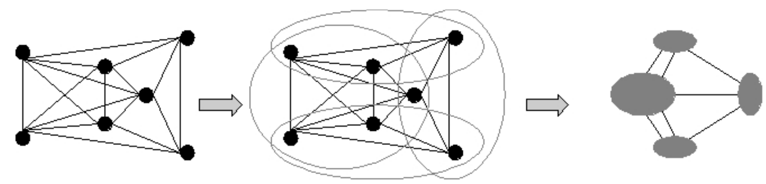
1. A vertex corresponds to a bank, and the edges between vertices correspond to bridges between banks.
2. A vertex corresponds to a bridge, and the edges correspond to banks shared by bridges.



It seems that the second graph appears frequently in classrooms (half the math teacher candidates constructed this graph in our experiment, and all the 20 secondary school students constructed this graph in one of our colleague's class!). The problem of finding a trail crossing the seven bridges then becomes:

1. "Finding a path in the graph that goes across each edge" that is to say "finding an eulerian path in the graph".
2. "Finding a path in the graph that goes across each vertex" that is to say "finding a hamiltonian path in the graph".

It is important to notice that there is a loss of information with the second graph, although it has more vertices and edges and thus might seem more "complete" than the first one. Indeed, whereas a trail in the town corresponds to a path in the graph, the converse is not true: a path in this graph does not necessarily correspond to a trail in the town. For instance, to which trail would correspond the path *a b c d* ? After crossing the bridges *a, b* and *c*, you are then necessarily standing on the bank *D*, and you could not cross the bridge *d*. To handle this, we have to add to the graph the information of *which banks* are shared by which bridges, by, for example, circling sets of bridges sharing the same bank. Then, to go from one bank to another one, one has to use a bridge belonging to the intersection of the corresponding circled sets. Finally, if we consider the circled sets as edges and the common points between two circled sets as vertices, then we get a new graph, which turns out to be exactly the one of model 1:



In addition, modelling can be an activity that emerges even when not posed explicitly in the problem. For instance, in the museums and graphs situation, almost all the pupils immediately claimed that "graphs are like museums" in week 2. The graph-museum correspondence was something very clear for them (see Figure 4), and when we asked them to what kind of museum would correspond the graph of Figure 5, they immediately replied "a museum with tunnels!". Four over five groups used their week 1 results about museums to solve the problems posed about graphs in week 2.

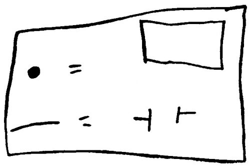


Figure 4: a point for a room, a line for a door

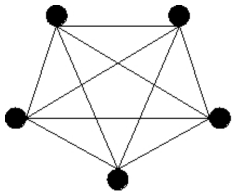


Figure 5: A non-planar graph

Moreover, even for the *representation* of the data, the graph model is of interest, since a graph is far more easier to draw than a museum map or a bridges and islands network. In the museums and graph situation, the learners produced a very large number of graphs to test their conjectures in week 2, whereas very few museums were drawn in week 1.

Conclusion

In this paper, we discuss some conceptions – namely *necessary and sufficient conditions*, the *proof*, and *modelling* – that appeared during experiments in class we carried out, related to the so-called Königsberg's bridges problem. A remarkable feature of these conceptions is that they seem to emerge independently of the age, the mathematical background, or the way the problem is presented. In that sense, we consider these conceptions as *transversal*. Thus, our work may be considered as an additional argument in favour of using discrete mathematics for "research problem" situations, and, more generally, for the learning of proof and modelling. Indeed, we observed a quick *devolution* of learners during our experiments, who addressed problematics that one may consider as lying at the very heart of mathematics, like *what is a proof?* and *what is a good model?* Interestingly, a careful reading of the seminal paper on the problem (EULER 1736) reveals that Euler himself might have encountered the same difficulties as those encountered by our learners, especially concerning the difference between necessary and sufficient conditions. Besides "research problem" situations, we also used the framework of a document study (posed as a written homework) followed by a debate for one of our experiments. We claim that this rather unusual (at least to our knowledge) framework is a very relevant one, especially with learners having a solid background in mathematics, which enables to study their conceptions in a way which is complementary to "research problem" situations. We noticed that the fact that our learners already knew the so-called "Euler's theorem" in graph theory was not an obstacle for the experiment. Indeed, the same problematics and difficulties that we observed during the "research problem" situations appeared during this experiment, both in the document study and during the debate. Whereas the resolution of Königsberg's bridges problem seem widely known and easy, even learners who are to know the problem and its solution may learn something about mathematics, for instance by reading Euler's paper and trying to understand and check his arguments. Thus, we claim that this problem is actually rich, and may be fruitfully used for educational purposes at different levels (from primary school pupils to undergraduate students or maths teacher candidates), with different approaches, from "research problem" situations to document studies and debates.

References

BIGGS N. L., LLOYD E. K., WILSON R. J. (1976) *Graph theory 1736-1936*, Oxford University Press.

BONDY A., MURTY U. S. R. (1976) *Graph Theory with Applications*, North-Holland.

CARTIER L. (2004) *Les graphes en Terminale ES - Etude didactique de l'introduction d'éléments de la théorie des graphes au programme de l'enseignement de spécialité en Terminale économique et sociale*, Master's thesis, Université Joseph Fourier, Grenoble.

COGIS O., ROBERT C. (2003) *Théorie des graphes : Au-delà des ponts de Königsberg – Problèmes, théorèmes, algorithmes*, Vuibert.

COUPY E. (1851) Solution d'un problème appartenant à la géométrie de situation par Euler, *Nouvelles annales de mathématiques* 10, pp 106-119.

DASGUPTA S., PAPADIMITRIOU CH., VAZIRANI U. (2006) *Algorithms*, McGraw-Hill.

EULER L. (1736) Solutio problematis ad geometriam situs pertinentis, *Commentarii academiae scientiarum Petropolitanae* 8, pp 128-140.

GRENIER D., PAYAN CH. (1998) Spécificité de la preuve et de la modélisation en mathématiques discrètes, *Recherches en Didactique des Mathématiques* 18(1), pp 59-100.

HIERHOLZER C. (1873) Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren, *Mathematische Annalen* 6, pp 30-32.

LABORDE C., PERRIN-GLORIAN M.-J. (Editors) (2005) *Educational Studies in Mathematics*, Volume 59, Springer.

LUCAS É. (1882) *Récréations mathématiques - Tome 1 - Les traversées. Les ponts. Les labyrinthes*, Gauthier-Villars, Paris.

ROLLAND J. (1999) *Pertinence des mathématiques discrètes pour l'apprentissage de la modélisation et de l'implication*, Ph.D. dissertation, Université Joseph Fourier, Grenoble.

ROUSE BALL W.W. (1892) *Mathematical recreations and problems of Past and Present times*, Macmillan, London.

SACHS H., STIEBITZ M., WILSON R.J. (1988) A historical note: Euler's Königsberg letters, *Journal of Graph Theory* 12, pp 133-139.

VERGNAUD G. (1991) La théorie des champs conceptuels, *Recherches en Didactique des Mathématiques* 6(1), pp 133-169.

WILSON R. J. (1986) An eulerian trail through Königsberg, *Journal of Graph Theory* 10, pp 265-275.

WITTGENSTEIN L. (1978) *Remarks on the foundations of mathematics*, MIT Press.