# Graph isomorphisms, matrices and a Computer Algebra System: switching between representations

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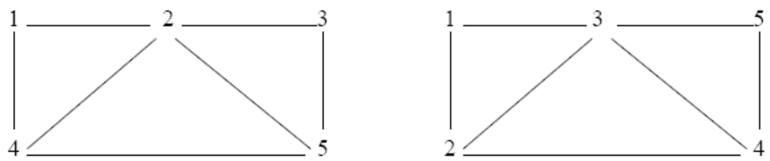
**Abstract**: We study classroom activities where prospective teachers meet a problem in Graph Theory, with an application of an advanced theorem in Linear Algebra. The support provided by a Computer Algebra System is analyzed, in particular with regard to the building of new mathematical knowledge through a transition from graphical to algebraic representation. The need for more than one representation is also discussed.

## I. Introduction.

An important problem in computational complexity theory is determining whether, given two graphs $G_1$ and $G_2$, it is possible to re-label the vertices of one graph so that it is identical to the other, or not. This re-labeling is called *a graph isomorphism* and we denote $G_1 \cong G_2$ .In simple words, two graphs are isomorphic if they can be represented with identical drawings. For example, see Figure 1: the permutation of vertices $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$ preserves the existence (resp. the non-existence) of an edge between vertices, whence ensures the fact that the two given iconic representations correspond to isomorphic graphs. A formal definition of a graph isomorphism can be found in Rosen's book (1999, p. 460).




**(a) First labeling** **(b) Second labeling**

**Figure 1: One graph, two label sets.**

*Adjacency matrices* are used to describe graphs in a computational way. For a given graph, label the rows and the columns of a square matrix $A = (a_{ij})$ by the vertices of the graph. For a non-oriented graph, $a_{ij}$ is the number of edges between vertices *i* and *j*. For an oriented graph $a_{ij}$ is the number of arrows from vertex *i* to vertex *j*. Thus, the adjacency matrix of a non-oriented graph is symmetric and for an oriented graph the adjacency matrix can be either symmetric or non-symmetric. Relabeling the vertices of the graph changes the adjacency matrix in the same way reordering the vectors of a basis of a *n*-dimensional vector space changes the matrix of a linear operator: the original matrix *A* and the new one *B* are similar, i.e. there exists an invertible square matrix *P* of order *n* such that $B=P^{-1}AP$.

Graphs, multigraphs (whether oriented or not) are defined as abstract objects, namely a pair of sets with a suitable property linking them (see Rosen 1999). We have here two presentations for a graph:

- The graphical presentation is visual/iconic (Lesser and Tchoshanov 2005) and acts as "stimuli on the senses" (Janvier et al. 1993).
- The other representation is algebraic. It is a symbolic representation enabling manipulations.

As noted by Lesser and Tchoshanov (2005), a single type of representation does not insure student learning and performance. In many occurrences, a graphical representation is used to "encode" more abstract properties. Here we will see that the symbolic-algebraic representation is a useful tool for the student to understand the graphical situation and to achieve a more profound insight.

In fact we replace a non easy problem in Graph Theory by a problem in Linear Algebra, not easier. To determine whether two given square matrices of the same order are similar is easy when both are diagonalizable. If they have the same eigenvalues, with the same respective multiplicities, then they have the same diagonalization, up to a re-ordering of the chosen eigenvectors. Suppose that diagonalizations of the matrices $A_1$ and $A_2$ exist and are given by $D = P_1^{-1} A_1 P_1$ and $D = P_2^{-1} A_2 P_2$, for appropriate invertible matrices $P_1$ and $P_2$, then $A_2 = (P_1 P_2)^{-1} A_1 (P_1 P_2)$ i.e. $A_1$ and $A_2$ are similar. If the matrices are not diagonalizable, similarity is harder to check. Of course, if one matrix is diagonalizable and the other is not, they are non similar. Note that the theorem sustaining the classroom activities is a "if ... then ..." theorem, not a "if and only if" theorem. If the graphs $G_1$ and $G_2$ are isomorphic, then their adjacency matrices have the same eigenvalues, but the converse is not true (see Cvetkovič et al. 1995, pages 61 sq.).

The volume of the computations increases very fast with the number of vertices of the graphs. Here a Computer Algebra System (CAS) reveals useful for technical assistance on computing. But not only for this assistance. Outsourcing the computations to the CAS may yield a better understanding of the mathematical situation and enhance understanding of older knowledge. "Technology can be used to compute, to reinforce, clarify, anticipate, or get acquainted with ideas, and to discover and investigate phenomena" (Selden, 2005).

## II. The study frame.

The Jerusalem College of Technology (JCT) is an Engineering School for High-Tech and Orot College is a Teacher Training College. In both institutions students learn a one-year course in Linear Algebra and an introduction to Graph Theory is given as part of a subsequent course in Discrete Mathematics. Matrix similarity belongs to the Linear Algebra syllabus. For various reasons, this topic has been taught at the very end of the course and quite no application to other fields of mathematics has been shown, beyond the fact that a basis change transforms the matrix of a linear transformation into a similar matrix.

Isomorphisms of graphs are an important topic in the syllabus. Conversations with colleagues teaching parallel courses revealed that students learn generally existence theorems related to degrees of vertices. Several textbooks do not mention more than this and the exercises are based either on the definition only or on such theorems about degrees of vertices (or in-degree and out-degree for directed graphs). Students are often reluctant to use adjacency matrices beyond writing the adjacency matrix of a given graph, or conversely drawing a picture of a graph whose adjacency matrix is given. "The computations are heavy", they say (generally in relation with the computations of the number of paths of a given length). For this reason, among others, the Derive software has been used in the classroom activities, in particular the Linear Algebra algorithms. Note that the Linear Algebra packages of other CAS can assist this activity, sometimes with specific outputs. More than two thirds of the students in the class had a good CAS literacy, as a result of two previous courses strongly based on CAS use. The other students had this opportunity to improve their knowledge and their know-how with regards to Derive.

The central topic of the activities is new for the students. It has to be introduced and developed explicitly by the teacher, using *strategic scaffolding*, one of the scaffolding categories detailed by Hobsbaum et al (1996); Anghileri (2006, p. 36) elaborates on this issue. The main characteristics are:

- A measured amount of teacher support;
- A careful selection of the tasks and of their difficulty level;
- Students' ability to build a mathematical meaning from the given tasks;
- Explicit strategies.

For the CAS assisted part of the work, we refer to Fischer's (1991) didactical principle of *outsourcing operative knowledge and operative skills*. Peschek and Schneider (2001) regard operative knowledge as a means to generate new mathematical knowledge (see also Peschek 2005). In fact they distinguish three *fields of competence*: basic knowledge, operative knowledge and skills, and reflection. In the following activities, the needed basic knowledge is matrix similarity, acquired at the end of the Linear Algebra course. Because of a lack of time in this course, the mathematics has been shown but almost not concretely manipulated. Students have now an opportunity to manipulate this knowledge in an applied situation. The operative skills are outsourced to the CAS. Most students (but not all of them) had already good operative skills for matrix computations using the CAS, including computation of eigenvalues and eigenvectors. During the sessions, they could improve these skills and discover new commands of the CAS. Moreover, new mathematical knowledge has been constructed, new CAS being part of it.

## III. Classroom activities with CAS.

We present here two classroom activities with a small group (15 students) of 20 year old teacher trainees. Their course in Graph Theory comes one semester after the course in Linear Algebra. This enables them to use eigenvalues and diagonalization of matrices in a situation very different of what has been met either in Linear Algebra or in other courses with some geometric flavor. The students were already used to switch from iconic representation to algebraic representation and from algebraic representation to iconic representation.

### 1. First activity.

Consider the graph with two different vertex labeling given in Figure 1. The respective adjacency matrices are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

`Operative knowledge:` using Derive's command **eigenvalues**, the students found that both matrices have the same five distinct real eigenvalues. Thus, the matrices $A_1$ and $A_2$ are diagonalizable, and for suitable eigenvector orderings, both matrices have the same diagonalization. It follows that the matrices are similar, whence $G_1 \cong G_2$ .

`Reflective thinking:`

Mina: This is not new; we knew already that the graphs are isomorphic!
Vered: So what?
Mina: Why did we do all this work?
Vered: We are now convinced that our way of working is right. Not?
Silence for a while. The second student sees that something still "disturbs" the first student. So she adds:
Vered: We see always a trivial example when learning something new. So we are really sure that the theorem is right. I will do the same thing when I"ll teach.

This remark was important for the teacher. It shows that Vered is aware not only of the new mathematical knowledge she is currently leaning, but also of the structure of the educative sequence.

**2. Second activity.**

We consider the two graphs shown in Rosen's book (1999, p. 461, example 10); see Figure 2. The vertices of the graph $G_1$ will be denoted by $u_k$ and the vertices of $G_2$ by $v_k$, $k = 1,..,8$ .

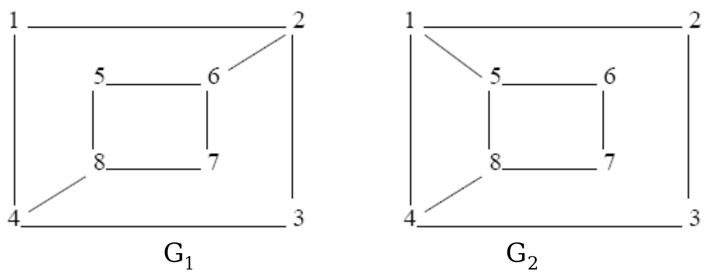



**Figure 2: Two non isomorphic graphs**

`Reflective thinking:`

Teacher: Let us check whether these graphs are isomorphic or not.
Vered: Easy! We check the degrees of the vertices.
Short silence, everybody computes.
Vered: These are the same degrees.
Teacher: So, what is your conclusion?
Leah: The graphs are isomorphic.
Short silence.
Hadas: Maybe not.
Vered: Why not?
Hadas: The degrees are not at the same place in the two graphs.
A couple of students, together: She is right!

The teacher asks for a clearer explanation of what happens. One student explains that the degree 3 vertices compose a connected subgraph in $G_1$ , but not in $G_2$ . This convinces the class that the two graphs are not isomorphic, but more than 10 students demand what they call "a stronger algebraic proof".

*Operative knowledge:*

Vered: Let's use matrices as we did before!
Teacher: Good idea, do it. Please write down the adjacency matrices.
The adjacency matrices of the two given graphs are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

With Derive's command **eigenvalues**, the students determine the eigenvalues of $A_1$ .
The output is: 0, -1, 1, $\frac{1}{2}+\frac{\sqrt{17}}{2}$, $\frac{1}{2}-\frac{\sqrt{17}}{2}$, $-\frac{1}{2}+\frac{\sqrt{17}}{2}$ and $-\frac{1}{2}-\frac{\sqrt{17}}{2}$.

*Reflection:*

Teacher: Any comments?
Short silence.
Shira: There are not enough.
Teacher: Not enough what?
Shira: Not enough eigenvalues. There are only 7.
Teacher: What did you expect?
Myriam: Eight.
Teacher: So, what happened?
Short silence.
Vered: There must be one double.
Teacher: Why?
Vered: The matrix is symmetric, it must have a diagonalization.
Teacher: Very nice. How can we know who is the double eigenvalue?
Short silence.
Yael: (with a short hesitation) how can we compute the characteristic polynomial?
Myriam: It's a determinant, there must be a command.
Teacher: Right. Who knows?

*Operative knowledge:*

A couple of students answer that the command is **charpoly.** The teacher recalls the syntax. With this command, the result is

$$P(\lambda) = \lambda^8 - 10\lambda^6 + 25\lambda^4 - 16\lambda^2 = \lambda^2\left(\lambda^6 - 10\lambda^4 + 25\lambda^2 - 16\right)$$

Vered: Here it is; it's 0.
Teacher: Vered, what is 0?
Vered: The double eigenvalue.
Myriam: How nice!!
Tehila: OK, but what do we do now?
Vered: The same thing with the other matrix.

The computation for $A_2$ is performed the same way, using Derive. The eigenvalues are

$$\frac{1}{4}\left(3+\sqrt{5}+\sqrt{22-2\sqrt{5}}\right), \frac{1}{4}\left(3+\sqrt{5}-\sqrt{22-2\sqrt{5}}\right), \frac{1}{4}\left(3-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right),$$

$$\frac{1}{4}\left(3-\sqrt{5}-\sqrt{22-2\sqrt{5}}\right), -\frac{1}{4}\left(3+\sqrt{5}+\sqrt{22-2\sqrt{5}}\right), -\frac{1}{4}\left(3+\sqrt{5}-\sqrt{22-2\sqrt{5}}\right),$$

$$-\frac{1}{4}\left(3-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right), -\frac{1}{4}\left(3-\sqrt{5}-\sqrt{22-2\sqrt{5}}\right).$$

*Reflection:*
Teacher: What do you see?
Shira and Vered: (at the same time) they are different.
Teacher: Different from what?
Shira: From $A_1$ .
Vered: We did it! The matrices are not similar.

### 3. Brief description of further activities.

Further work and activities have been done with the same class. After the second activity, students received homework assignments. The next meeting took place one week later. A central task was to show that two given graphs with the same number of vertices and the same number of edges are non isomorphic. For this, almost no teacher intervention was necessary. Another task was devoted to understanding the non-reversibility of the theorem described at the end of the first section (see Cvetkovič et al. 1995, pages 61 sq.); its description could be too long for this paper.

The next step in the same meeting consisted in turning students' attention towards similar situations, either with oriented graphs or with multigraphs. Definitions can be found in Rosen's book (1999). The class dealt with the new situation using the same algebraic and technological tools, but in a different algebraic situation. The same CAS commands were used as in the previous sessions.

The third meeting was devoted first to oriented graphs because they may provide non-symmetric adjacency matrices, i.e. matrices which can be non diagonalizable. Then the students had to study a pair of non isomorphic graphs having the same set of eigenvalues. The goal of this last example was to convince the students that the whole study relied on a one-way theorem: if two matrices are similar, then they have the same set of eigenvalues, but the converse is not true.

After the third meeting, the teacher had a quite informal discussion with the students. He asked for remarks about the CAS assisted work. Here are a few excerpts from the discussion between students (the first one was not previously quoted):
Student A: I'm sure that I would not have worked out all these examples by hand.
Student B: And so? You would not have learnt this?
Student A: No, I would have waited to see the answer from somebody else.
Student B: And so you would not have learnt the topic!
Student A: (hesitating) Maybe you are right, …, not so well.

## IV. The role of the CAS; different representations.

In the first activity, the teacher chose an example where the isomorphism between the graphs is trivial. The graphical display itself shows an appropriate invertible mapping between the sets of vertices. This enabled the students to discover how to work, in a situation where they have control on the results. Vered expressed this clearly.

The second activity followed in the same session. The students were more independent from the teacher. He helped somehow with passing from one step of reflective thinking to the next one, or with providing some new operative knowledge, such as an appropriate command of the CAS. The teacher's support was gradually faded; it was limited to questions, and reflection and interpretation were made by the students.

Teacher's support has been gradually removed during the third activity. At the end all the students but two were totally independent of teacher assistance. This has been checked with an assignment which included the study of one pair of non oriented multigraphs and of one pair of oriented multigraphs. Finally, the educative segment has been spread over a little more than two weeks, and gradually developed, according to Anghileri's request (2006). The strategy has been made clear already from start:

- A progressive choice of examples: non oriented graphs, in order to have benefit of the theorem on the diagonalizability of symmetric matrices, then non oriented multigraphs and oriented graphs for which the theorem does not apply.
- Translation into notions from Linear Algebra and use of a CAS.

The usage of the CAS enabled the students to improve their insight into the topic. Considering only iconic representations of graphs does not yield enough insight into the concept of an isomorphism of graphs beyond simple examples, as suggested by Leah's reaction, and more by Student A in the last discussion. The matrix representation and its companion algebraic tools provide a possibility to have a more profound insight. "In some cases the current representations may prove an obstacle to the full development of a concept" (Ferrari 2003). Out of the record, students claimed that iconic representation is more readable for them, but others said that they felt more comfortable with matrix representation, as "they can do computations". Despite the fact that matrix representation is more abstract than the graphical one, it opened the way to new mathematical knowledge, through manipulation both of old knowledge and of an algebraic-computational tool. A great diversity of situations could not have been presented using graphical representation only (see Lesser and Tchoshanov 2005). Actually we may view the working sequence as a double outsourcing:

a. Outsourcing from the iconic representation to the matrix representation, according to Peschek (2005), as "one abstracts relationships from the (reference) context and presents them with symbols, thus outsourcing the problem in the formal-operative system of mathematics".
b. Outsourcing operative knowledge to the computer.

In another direction, the CAS provided the help ``for reasoning by fostering the development of ... experimental reasoning style" (Sinclair et al. 2006). This appears through the intertwining of reflective thinking and application of operative knowledge during the sessions. A difference appears with the human support: not only the CAS assistance does not fade with time, but the new computing skills become an integral part of the new mathematical knowledge.

In the second activity, different CAS may give different outputs when displaying the eigenvalues. For the given square matrices of order 8, Derive gives seven different eigenvalues, inviting the student to understand that one of them must be a double eigenvalue. Note that other packages may give a more detailed output, including the multiplicities of the eigenvalues. We have here an example of the *double reference* evoked by Artigue (1997, page 152): on the one hand, the computer ``understands" the input in a way which can be different from the students' intention, on the other hand the mathematical meaning of the output can be different of what the student expects when he/she writes the same thing. See also (Lagrange, 2000).

Students working with a CAS become progressively acquainted with swapping between various representations: algebraic, numerical and graphical. For a given object, different representations can be provided by the CAS itself. Functions of one real variable are a well documented example, with numerical representation (a table of values), graphical representation and generally algebraic representation (a "closed form" such as $f(x) = \dots$ ) . The main problem is developing students' ability to link representations; see Pierce (2001).

The activities performed by the students have a different aspect. First of all, the link is not oriented from an algebraic representation towards a graphical one as in most problems on one-real-variable functions, but in the other direction. Graphs are given by graphical representations and the representation used for checking the existence of an isomorphism between two graphs is purely algebraic. Technically, this can be viewed as a trivial task, but from a conceptual point of view, it is not. The students' hesitations reveal their ability level to deal with a matrix representation instead of a graphical one. Previous working sessions revealed the difficulty for students to link matrices to graphs and graphs to matrices (including oriented graphs, i.e. links towards non-symmetric matrices) but helped with removing the obstacles. The CAS provided assistance, and students showed increasing operative knowledge. Consolidation of this knowledge has been showed in the last session (not described here).

The activities revealed also that despite the fact that for symmetric matrices and their diagonalization, the students did not achieve the routinization mentioned (and requested) by Artigue (1997). In Section II, we mentioned the lack of time at the end of the Linear Algebra course, which provoked a shortage in solved examples. Even for low dimensions, a lot of computations is needed, looking for eigenvalues and eigenvectors, inverting matrices (this operation was not necessary here). Hand computations are very unilluminating and both educators and students are reluctant to do them. The students had here an opportunity to practice computations of eigenvalues and eigenvectors, and sometimes of the diagonalization of a matrix. Moreover they had an opportunity to deal with a concrete problem involving these tools. The CAS was a facilitator, making examples of higher dimension possible to treat, thus enabling students to acquire an extended operative knowledge, and at the same time more mathematical insight.. The CAS has not been used as a black box, but rather as an assistant in a process of *reasoned instrumentation*. We meet Elbaz-Vincent's requirements (2005) about ``the necessity of developing specific classroom activities and specific exercise sheets, ..., showing clearly the value of the CAS either as a platform for experimentation or as an assistant ..."

CAS assisted work had another side effect. For non isomorphic graphs, the following cases can appear:

a. The adjacency matrices have different characteristic polynomials.

b. The adjacency matrices have the same characteristic polynomial, whence the same eigenvalues with the same multiplicities, but one of the matrices is diagonalizable and the other one is not.

At the beginning, the command **eigenvalues** has been used without reference to the characteristic polynomial. The necessity to obtain more information, and to know how to interpret the output, has revealed the necessity of another command. During the activities, a black box has been opened and examined.

The assistance provided by the CAS is useful only if the students are able "to plan correct operations and to interpret results intelligently" (Fey 1990, quoted by Pierce 2001). Two remarks made by students emphasize this issue:

(i) In the second activity, Shira's remark on the number of eigenvalues is important. It has been provoked by Derive's output, where the eigenvalues are given, without mention of their respective multiplicities.

(ii) The meaning of Vered's claim ``we did it" is non trivial. She noted that, despite the regular usage of a CAS to provide explicit numerical results, this time the actual eigenvalues of the matrices were quite irrelevant. The important issue was the comparison between the two sets of eigenvalues. Vered has understood that the fact that the eigenvalues are not the same is the important issue.

There are not so many opportunities to convince students that the precise values of computation results are not the only interesting output. In this study, we found a couple of occurrences where the precise eigenvalues were not interesting. The point was in the comparison between the sets of eigenvalues. The outsourcing of the computations has an effect beyond the computations themselves. The CAS assisted activities described in section III are an example of the claim by Cuoco and Goldenberg (1996): "...we are talking about using technology in support of the hard thinking, not for performing the low-level details". More than acting as a calculator, the CAS worked here as an assistant to reflection.

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