Discrete mathematics: a mathematical field in itself but also a field of experiments

A case study: displacements on a regular grid

Cécile Ouvrier-Buffet (Paris 12 IUFM de Créteil, Paris 7 DIDIREM, Maths à Modeler)

Introduction

The main feature of my mathematical and didactical research, exemplified in this paper with a situation in discrete mathematics, concerns 'definitions construction processes'. Although definitions construction admittedly has a place in mathematical research, this object (that is defining processes) is rarely studied for itself. The core of my research has the following guiding idea: to map the field provisionally with definitions serving as temporary markers for concept formation. I therefore had to work out a theoretical framework through epistemological, didactical and empirical research in order to characterize definitions construction processes (Ouvrier-Buffet, 2003, 2006). My experiments were conducted in discrete mathematics with the following concepts, which are of a different nature: trees (a known discrete concept, graspable in several ways), discrete straight lines (a concept which is still at work, for instance in the perspective of the design of a discrete geometry) and a wide study of properties of displacements on a regular grid. I have chosen to develop this last point for two reasons. Firstly, the explanation of this kind of situation will bring some answer elements to one of the questions of this TSG: "How can discrete mathematics contribute to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge?" A mathematical work on ("linear") positive integer combinations of discrete displacements actually mobilizes skills such as defining, proving and building new concepts. Secondly, it leads us to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation. Therefore, a new question emerges: discrete mathematics represent a mathematical field in itself (as well as linear algebra for instance), but can they also be a field of experiments in order to question at the same time the skills, knowledge and concepts involved in other mathematical fields as well?

In this paper, I intend to present the mathematical discrete issues brought by the situation of discrete displacements on a regular grid and to raise the problematic of the mathematical links between the discrete problem and the continuous problem.

Presentation of the situation

General problem: set of displacements on a regular grid ($\mathbb{Z}^2$)

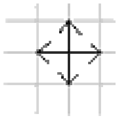
Let G be a discrete regular grid. This grid can be squared or triangulated for instance. For the rest of this article, G is a squared regular grid. A "point" of the grid is a point at the intersection of the lines. Let A be a starting point. An **elementary displacement** is a vector with 4 **positive** integer coordinates (it can be described with the directions: up, down, left and right). For instance "2 squares right and 3 squares down" is an elementary displacement which can be represented with a vector. A **displacement** is a **positive** integer combination of elementary displacements. We will write $a_1d_1 + a_2d_2 + \ldots + a_kd_k$ for an integer combination of k elementary displacements where a_i are **natural numbers** and $1 \leq i \leq k$.

The general problem is: let E be a set of *k* vectors with integer coordinates. Starting from a given point, which points of the grid can one reach using positive integer combinations of vectors of E?

Mathematical study

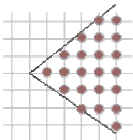
1) How to reach all the points of the grid?

A set of displacements which allows reaching all the points of the grid exists. The four elementary displacements represented below obviously do it.

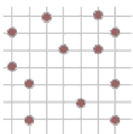


Now, can we characterize all the sets of displacements which allow us to reach all the points of the grid? We have to work on two different properties simultaneously:

- the "density": All the points of a zone of the grid are reached.



- and "a little bit everywhere": Let P be a point of the grid. There exists a reached point, called A, "close to P", i.e. such as the distance between A and P is bounded[1] (for every P, independently of P). We will call this property "ALBE".



We can reach all the points of the grid when these two properties ("density" and "ALBE") are verified simultaneously. These properties imply the definition of "generator set".

2) Reciprocal problem and minimality

Let E be a set of elementary displacements. What points can one reach with E? When the set of reached points is characterized, a new question emerges: is it possible to remove an elementary displacement of E without changing the reached points? This is a question about the minimality of the E set. E is called **minimal** when removing one of its elementary displacements modifies the set of reached points. With this definition, how to characterize a minimal and generator set of displacements? Furthermore, are the minimal and generator sets of displacements **minimum** too i.e. do they have the same cardinality?

3) Paths and different paths

Let E be a set of k elementary displacements written as $d_1, d_2, \ldots, d_k$. What can we say about the paths from the A fixed point to the B reached point? We call path from A to B an integer combination of elementary displacements of E. A path can be described by a k-tuple $(a_1, a_2, \ldots, a_k)$ where a_i, for $1 \le i \le k$, are the integer coefficients of this combination.
Two paths from A to B are called different if and only if the k-tuples characterizing them are different. Note that the order of the elementary displacements does not interfere because of the commutativity of displacements.

[1] Distance on the grid or every other Euclidean distance.

Then, we can interrogate the relation between the cardinal of the paths from A to B and the minimality of E: *when there are (at most) two different paths, is it possible to remove an elementary displacement in E?*

The answer is no: the study of that is a difficult one, even if we limit the study to $\mathbb{N}$. Here is a counter-example on the discrete line. Let E be a "displacement" composed by 2 squares to the right and 3 to the right, i.e. E is composed by the natural numbers 2 and 3, and we look at the numbers which can be generated by 2 and 3. With the displacements of E, we can reach 11 in two different ways: either with $4 \times \mathbf{2} + 1 \times \mathbf{3}$, or with $1 \times \mathbf{2} + 3 \times \mathbf{3}$. But we cannot remove 2 or 3 from E otherwise 11 will not be reached. Then, E is generator and minimal for 11. It can lead us to the famous Frobenius problem (Ramirez-Alphonsin, 2002) that we will not develop here.

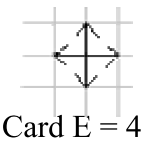
We notice that the existence of several paths does not necessarily imply the non-minimality of E. Then we have to consider different kinds of E sets:

- There is no uniqueness of the path for one point at least i.e. there exists at least one point which can be reached with at least two different ways: it does not imply that E is non-minimal;
- Every point of the grid can be reached at least in two different ways. We call this property "**redundant everywhere**". Thus, the E set is non-minimal: this is the case when an elementary displacement of E is an integer combination of other elements of E;
- Every point of the grid can be reached in only one way (uniqueness of the path): we call this property "**redundant nowhere**". The E set is clearly minimal.

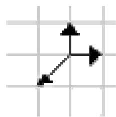
4) Minimal generator sets of $\mathbb{Z}$ and their cardinalities

The minimal generator sets can have different cardinalities.

For example, you can see below a minimal generator set with 4 elements and another one with 3 elements: with both of them you can go everywhere on the grid, that is to say "ALBE" and with "density".



Card E = 4



Card E = 3

We can succinctly study this specificity of the discrete case with the integers.

In order to build a set of minimal generator elementary displacements on $\mathbb{Z}$, we have to use coprime numbers (i.e. gcd of them is equal to 1). Thus, the "density" property is true for natural integers (Bezout's theorem[2]). Some of these coprime numbers should be negative in order to go "a little bit everywhere" (a little bit to the right and a little bit to the left). For example, if we want to generate $\mathbb{Z}$ with 4 integers, we build 4 natural numbers which are coprime as a whole (for instance $2\times3\times7$, $3\times5\times7$, $2\times3\times5$, $2\times5\times7$ i.e. **42, 105, 30** and **70**). Then we can reach 1 (according to Bezout's theorem) that is to say we can go with density on $\mathbb{N}$. Now if we take one of these numbers as a negative one, we can go "a little bit everywhere" and we get: E = {**42; 105; - 30; 70**} **is a generator of $\mathbb{Z}$.** So we can build several sets of minimal generator displacements with different cardinalities. Another example: E = {**1**; **-1**} and F = {**2** ; **3** ; **-6**} are generator and minimal, card(E) is 2 and card(F) is 3.

Then, we have the following theorem:

[2] Bezout's theorem: integers $a_1, a_2, \ldots, a_n$ are coprime as a whole if and only if there exists integers $u_1, \ldots, u_n$ such as $u_1 a_1 + \ldots + u_n a_n = 1$.

Theorem: there exists, in $\mathbb{Z}$, sets of minimal generator elementary displacements with *k* elements, *k* being as big as one wants.

Therefore, the cardinality of sets of minimal generator elementary displacements of $\mathbb{Z}$ is not an invariant feature. However, the study of the generation of integers has showed that this problem is mathematically closed for $\mathbb{Z}$. The reader can consult the wider and more complex Frobenius Problem (Ramirez-Alphonsin, 2002).

We will show that the problem is not mathematically closed in $\mathbb{Z}^2$, by proving that we can build minimal generator sets with as many elementary displacements as we want.

5) Construction of sets of minimal generator elementary displacements, in $\mathbb{Z}^2$, with *k* elements

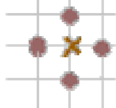
We call E_k the set of all generator displacements with *k* elementary displacements. We want to generate all the points of the regular grid. A starting point is given. The study of the "generator" and "minimal" properties on a discrete grid is more complex than on $\mathbb{Z}$: that is the reason why I choose to study the small values of *k* first.

Study of E_2

No set of two elementary displacements allows reaching all the points of the grid. Indeed, the property "ALBE" is never verified.

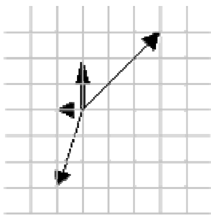
Study of E_4

If we consider the grid with the two directions "horizontal" and "vertical": in order to reach all the points of a horizontal straight line, we have to reach the points 1 and -1. A starting point is given (a cross marked in the figure below). The same idea applied to the vertical straight line crossing the starting point leads us to conclude that if we can reach the "4 cardinal points". Therefore, it is possible to reach all the points of the grid.



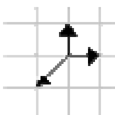
What happens with a horizontal or a vertical straight line ($\mathbb{Z}$) is described above.

Then, we can generate sets (cardinality = 4) of minimal generator elementary displacements with horizontal and vertical displacements. The reader can try to find different kinds of sets of minimal generator elementary displacements with 4 elements. Let us give another example.



Study of E_3

There are sets with 3 minimal generator elementary displacements. See the figure below for instance.



We could explain this set by considering the two properties characterising "generator": going "a little bit everywhere" and "density".

Study of E_5
Keeping in mind the horizontal/vertical representation, we can build sets of minimal generator elementary displacements with 5 elements: take two horizontal elementary displacements generating a horizontal straight line (with two prime numbers: a negative one and a positive one) and take three vertical elementary displacements generating a vertical straight line. In order to grasp a set with 5 minimal generator elementary displacements, we have to choose these three vertical displacements carefully (we do not want to remove a displacement, otherwise we loose the minimality!). For instance: 2, 3 and -6.

Generalization to E_k
The study of the first cases E_k, $k = 2, \ldots, 5$, leads us to a theorem of existence.

> **Theorem: there exist, in $\mathbb{Z}^2$, sets of minimal generator elementary displacements with *k* elements, *k* being as big as one wants.**

Indications for the proof
One constructs a set of horizontal minimal generator elementary displacements with (k-2) elements in order to generate $\mathbb{Z}$ (see above) and to add two vertical elementary displacements in order to go everywhere by translation ☺.

But, *k* being given (as big as one wants), we do not know how to construct all the sets with *k* minimal generator elementary displacements.

6) **How to prove that a set of elementary displacements is generator or minimal?**

The study above brings us two techniques to prove that an E set of elementary displacements is generator.
The first consists in proving that the elementary displacements of E allow going "a little bit everywhere" and with "density". The second consists in generating the points of a known generator system (such as the four cardinal points in E_4).
In order to prove that an E set of elementary displacements is minimal, we need to find out if it is possible to remove a displacement without changing the set of points generated by E. The study of the number of different paths from one given point to a point generated by E can lead to the removal of one elementary displacement. But several cases have to be considered, as seen above.

An experiment with freshmen

The first idea I had was that a linear combination can be materialized by a displacement on a regular grid. So I chose an activity with discrete displacements which contains Harel's Necessity Principle[3] (Harel 1990, 1998) in order to define and to circumscribe the concepts of generator, minimality etc.

In vector space, the notions of generator and dependence are highly correlated. The lack of formal definitions of these notions in a specific situation may allow an activity of *definition-construction* (Ouvrier-Buffet, 2006). This activity is decontextualized with regard to classical

[3] G. Harel explains the Necessity Principle translating into three steps:
1- Recognize what constitutes an intellectual need for a particular population of students, relative to the concept to be learned.
2- Present the students with a problem that corresponds to their intellectual need, and from whose solution the concept can be elicited.
3- Help students elicit the concept from the problem solution.

introduction of concepts in linear algebra. It is an open problem, which the students do not know. The concepts of generator, minimality but also dependence and basis can be studied. In this situation, the concepts are close to one another but not exactly the same as those in linear algebra. Indeed this activity brings more concepts than in a vector space (for example, the notion of minimality). We stress out the fact that the linear algebra is **not** the model for the situation of displacements. Linear algebra brings well-known obstacles, in particular with its definitions and a unifying formalism. So this explains the necessity of a "decontextualization" in order to give an access to the mathematical problematic. This decontextualization in discrete mathematics allows a work on properties which are co-dependant in the continuous case.

As seen in the mathematical study, the situation suggests an activity on the definition of "different" paths, but also the definition of generator, minimality, density and "a little bit everywhere". The students were induced to define besides being challenged to discover an answer to the "natural" questions: how can we reach each point of the regular grid? What does it mean? Does a minimal set of displacements exist in order to go everywhere? Furthermore, I assume that the notion of generator should come naturally contrary to the notions of dependence and minimal generator (basis).

Presentation of the experiment

This activity was carried out with a class of students in sciences (first university year). It consisted into three problems.

- ***General instructions***

Today we will take an interest in the displacements on a regular squared grid.
For each problem, we will use a set of displacements and we will choose a starting point (called A).
For each problem, we will ask ourselves the following questions:

1- *Starting from point A, which points of the grid can we reach?*
2- *Let's take another point, called B. Can we reach B from A? If so, are there different paths to do it? What does "different" mean to you?*
3- *Can we remove one or more displacements? If so, what are the consequences?*

- ***Problem 1**:*

  - *The authorized displacements are:*

d_1: 2 squares to the right and 1 square up.
d_2: 3 squares to the left and 3 squares down.

  - *B is 3 squares right and 3 squares down from A.*

- ***Problem 2**:*

  - *The authorized displacements are:*

d_1: 2 squares to the right and 3 squares up.
d_2: 5 squares to the left and 2 squares down.
d_3: 5 squares to the right and 3 squares down.
d_4: 1 square to the right.

  - *B is 2 squares right and 2 squares down from A.*

- ***Problem 3:***

  - *The authorized displacements are:*

d_1: 3 squares to the right and 3 squares up.

d_2: 2 squares to the left.
d_3: 1 square to the left.
d_4: 1 square to the left and 3 squares down.

- *B is one square right and 6 squares up from A.*

Brief mathematical analysis of the experiment

Each problem plays a specific role in the devolution of the situation and of the definitions process.
The first one allows students to be involved in the situation, to assimilate the rules of the displacements and the mathematical questioning. Students should conjecture that "in order to generate all the points of the grid, more than two elementary displacements are required".
In the second problem, we can remove one "good" displacement (well picked) without changing anything to the generator aspect. The notions of dependency and redundancy can emerge here. This problem leads to the proof that "three elementary displacements can be enough to generate all the points of the grid". But, after this, we do not know if all the generator systems are made up of three elements.
The main goal of the third problem is to demonstrate that "three displacements are not always enough". In fact, the set presented in this last problem features four displacements and is generator and minimal. So we can reach the following result: the dimension theorem (proved in a vector space) is false in the discrete case.
The reader is advised to actually solve these three problems for a better understanding of the analysis.

Presentation of some experimental results

I will expose a complete analysis of students' procedures during the Congress, exploring the concept formation and the perspectives that the situation of displacements offers to other fields of mathematics. But let me briefly outline some experimental results.

The situation of displacements allows a work on mathematical **objects** (displacements, paths) graspable through a basic representation close to that of vectors. The main difficulty lies in the fact that **properties** have to be defined (generator, independency, redundancy, minimality). Indeed, the objects we work with do not need to be explicitly defined at first: we have to focus on properties, to characterize and to define them.

These specificities of the situation of displacements partially explain why the students did not engage in characterizing mathematical properties. Indeed, only some *zero-definitions* (in Lakatos's sense, see Ouvrier-Buffet 2006) were produced but they did not evolve into operational definitions. Nevertheless, a "natural" definition of "generator" (i.e. "to reach all the points of the grid") has been produced and has been transformed into an operational property ("to generate four points or elementary displacements"). Furthermore, I have identified two *definitions-in-action* (in Vergnaud's sense): one for "generator minimal" and one for "minimal set".

The presence of *definitions-in-actions* proves that students can not stand back from the manipulated objects: students stayed in the action, in the proposed configurations. Their process did not move to a generalization which would have allowed a mathematical evolution of *zero-definitions* or *definitions-in-action*. A plausible hypothesis is that this *distance* (between manipulation and formalization, formalization merely a first step, not a complete theorization) is too rarely approached in the teaching process. It goes along the lines of previous epistemological and didactical results which conclude that formalism is a crucial obstacle in the teaching of linear algebra.

The didactical analysis of the productions of the students is very difficult. In fact, the dialectic involving definition construction and concept formation is useful to understand the students' procedures and their ability to define new concepts in order to solve a problem. To understand how concept formation works implies exploring the wide field of mathematical definitions considered as concepts holders. That will be discussed during the Congress using a theoretical framework specific to definitions construction (see Ouvrier-Buffet 2006).

Bibliography

Harel, G. (1990) Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. In *International Journal Of Mathematics Education in Science and Technology* n°21, 387-392.

Harel, G. (1998) Two Dual Assertions: the First on Learning and the Second on Teaching (or Vice Versa). In *American Mathematical Monthly* 105, n°6, 497-507.

Lakatos, I. (1961) *Essays in the Logic of Mathematical Discovery*, Thesis, Cambridge (University Library).

Lakatos, I. (1976) *Proofs and refutations*, Cambridge, CUP.

Ouvrier-Buffet, C. (2003) *Construction de définitions / construction de concept : vers une situation fondamentale pour la construction de définitions en mathématiques.* PhD Thesis. Laboratoire Leibniz, Université Joseph Fourier – Grenoble. France. Web Available http://tel.archives-ouvertes.fr/tel-00005515/en/

Ouvrier-Buffet, C. (2006) Exploring Mathematical Definition Construction Processes. *Educational Studies in Mathematics*. Volume 63, n°3, 259-282.

Ramirez Alfonsin, JL. (2006) *The Diophantine Frobenius Problem*. Oxford University Press.

Vergnaud, G. (1991) La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques* n°10/2.3, 133-169.

Vinner, S. (1991) The role of definitions in teaching and learning of mathematics. In Tall, D. (Ed.) *Advanced Mathematical Thinking*. Dardrecht: Kluwer.