Teaching of Discrete Mathematics at Advanced Level in Singapore: Teachers' Perspectives

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Introduction

Although Singapore has been lauded for its performances in the TIMSS (Singapore Ministry of Education, 2004) educators in Singapore continue to be concerned about the slack with regards to the development of creativity and disciplinary talent. While it is accepted that the mean achievement of Singapore students is like a high mountain range, it is acknowledged that Singapore also needs even higher peaks of achievement which are at the moment few and far between (Shanmugaratnam, 2004). As examples, in mathematics, Singapore has had only one International Mathematical Olympiad gold medalist, and at the highest levels, Singapore has yet to produce a Fields medalist.

To the end of always fine-tuning the education system to meet the changing needs of the nation and the world, the Singapore Ministry of Education conducts a comprehensive curriculum review for each subject about once every five years. Subsequent to the Report of the Junior College/Upper Secondary Education Review Committee in 2002, the Singapore Education Ministry announced the New 'A' Level Curriculum for pre-university in 2006 (Curriculum Planning Development Department, 2006). The new curriculum aims to provide students with life skills, knowledge skills and content-based subject knowledge. In the new curriculum, subjects have been revamped as Higher 1 (H1), Higher 2 (H2) and Higher 3 (H3). H1 subjects serve two purposes. They enable students to follow a subject of interest in which they do not intend to specialise or to acquire foundational knowledge in a subject area that will support their future studies at university level. H2 subjects are equivalent to the standard of A-Level subjects prior to 2006. Finally, H3 subjects are for students to pursue a given subject in which they are passionate about and keen to excel at a higher level.

In our paper, we shall discuss the implementation and motivation for including two Discrete Mathematics topics into the H3 syllabus and the training of teachers to teach these topics in their schools. We will also present the results of a survey of teachers' perspectives on the implementation of more advanced Discrete Mathematics into pre-university education.

Inclusion of Discrete Mathematics

The number of curriculum hours for H3 subjects ranges from 112 to 210, depending on the H3 subject. A student may take a maximum of two H3 subjects. The average number of hours schools allocate for H3 mathematics is 120 over an academic year of 40 weeks (10 weeks per term and 4 terms per academic year).

Students have to take H2 mathematics (for foundational knowledge) in order to take H3 mathematics since the H3 programme is intended to be more advanced and specialized.

Schools are given the autonomy to select students for H3 mathematics for the reason that they know the students better.

For mathematics, H3 is available in a variety of modes:

(a) as modules currently available in one of the universities and taught by university lecturers in the university;
(b) as modules specially tailored by another university and taught by the university lecturers in the schools; and,
(c) as modules taught by teachers in their own school or junior college, guided by a syllabus provided by the Singapore Examinations and Assessment Board (SEAB).

Since the number of places available for options (a) and (b) are limited, the majority of students are enrolled in option (c). Option (c) is of interest to us as it involves the implementation of more advanced mathematics within a school's capabilities. Of interest also to TSG15 Discrete Mathematics education is that two out of the three optional topics specified by the H3 mathematics curriculum are Graph Theory and Combinatorics. This H3 syllabus prescribes one compulsory topic, Applications of Differential Equations, and three options, Plane Geometry besides the earlier mentioned, two of which will be chosen by the student[1].

Motivation for Discrete Mathematics

It is commendable that the new curriculum aims to reflect the nature of the discipline and current development in the field. Its FAQs on the new curriculum (Curriculum Planning Development Department, 2006), gives the rationale for a subject to be offered at H3 as allowing for students, who are already taking the subject at H2 level, to "study the subject at greater depth or in a more specialized area" and for the H3 programmes to remain responsive to new advances in the field.

We interviewed Professor Lee Peng Yee, an eminent mathematician and mathematics educator in Singapore who was in the curriculum review committee for mathematics. He gave the reasons for selecting two Discrete Mathematics topics for H3 as the following:

- Its richness and accessibility.
- Various applications.
- Rigour in mathematical reasoning and writing.
- Training ground for mathematics and problem solving in context.
- Worldwide trend in relation to the topics of patterns, counting and probability making inroads into secondary school curricula.

[1] Readers interested in syllabus for H3 Mathematics can access it at http://www.seab.gov.sg/SEAB/aLevel/syllabus/2008_GCE_A_Level_Syllabuses/9810_2008.pdf

Implementation of H3 mathematics in the schools

The Singapore Ministry of Education is cognizant of the need to equip teachers with the necessary content-knowledge and pedagogy to implement the new syllabus (Shanmugarathnam, 2003). To this end, the Mathematics and Mathematics Education Department at the National Institute of Education in Singapore was engaged to provide specialized training for the H3 Mathematics. The training for Combinatorics and Graph Theory each consists of two series of 12-hour workshops. To date, about 100 teachers have undergone this training.

It is important to find out how H3 Discrete Mathematics is being implemented in the schools. We first approach this issue by surveying the teachers who are teaching H3 Discrete Mathematics. In future research, we will seek the views of students and administrators.

Description of the Study

With the approval from the respective Heads of Mathematics Department, a survey was conducted for the teachers from four, out of seventeen, Junior Colleges in Singapore.

The survey comprised questions seeking information, for each of the two topics, about (i) the teachers' beliefs about Discrete Mathematics, (ii) the teachers' beliefs about the students who are offering H3 mathematics; (iii) their competency and fondness of Discrete Mathematics; (iv) the relative ease and difficulty, and their students' relative fondness of the four different options of H3 mathematics.

A summary of the breakdown of the teacher participation rate and H3 student enrolment of the four Junior Colleges is shown in the following table. The four Junior Colleges were coded A, B, C and D.

Table 1

Teacher and student details of the four Junior Colleges

Junior College	**No. of teachers involved in Discrete Maths**	**No. of teachers who responded to survey**	**No. of students taking H3 in 2007**	**No. of students taking H3 in 2008**
A	8	1	6	0
B	8	8	149	198
C	4	2	18	24
D	2	2	28	35

The number of years that the teachers have taught in the junior college system ranged from 2 to 10 years, with a mean of 4.9 years. The teachers had at least a Bachelor's degree in Mathematics, with 6 having an Honours degree and 5 a Master of Science. The number of Discrete Mathematics courses read in the university ranged from 3 to 10, with a mean of 4.3.

Caveat: The findings presented and observed in the sections following should be interpreted with caution because of the small number of respondents. Also, not all teachers answered all the items because some were not relevant to them, such as when they had not attended a particular course or taught a particular topic.

Results and analysis of the survey

I Training Workshops

To prepare teachers to teach H3 Mathematics, in particular on Graph Theory and Combinatorics, a series of mathematics training workshops was jointly organized by the Singapore Ministry of Education and the National Institute of Education. The teachers were asked to rate, from 1 (Very Useless) to 5 (Very Useful), the usefulness of the workshops that they had attended. The Combinatorics workshops had averages of 4.0 and 4.1 while the Graph Theory workshops had averages of 3.6 and 3.8. These values indicate that in general, the teachers thought that the workshops were useful, particularly in providing them with content knowledge. In their own words,

> "The courses were generally very good at providing a strong and rigorous foundation to Graph Theory in terms of content knowledge."

> "Most of the current H3 courses aim at disseminating the content to the teachers which are useful in helping us to recall the relevant content learnt in our university days."

In the same survey, we asked the teachers to suggest ideas for future workshops on Discrete Mathematics, now that they have taught the subject. Their comments reveal a need for future workshops to provide instructional strategies and address pedagogical concerns.

Setting and marking test and examination questions can be challenging; in the words of one teacher:

> "For me, setting test and exam questions is the most difficult because during lessons, we try to expose the students to as many questions as possible so we run out of questions! Maybe more on setting questions. Marking Graph Theory is also rather challenging as different tutors expect different degrees of rigour. Solutions for Graph Theory are often "essay" type of answers."

This concern is supported by the findings from the survey (reported below).

Other teachers are keen to learn ways of motivating students to learn the subject, as revealed by comments:

> "It'll be good to give some real life applications of combinatorics at a level which the students will be able to understand."

> "How to teach the topic, for example the motivation, interesting and related result, flow of thought ..."

> "Class projects may be fruitful as Combinatorics is rich in applications."

There are pedagogical concerns that future workshops should address:

> "The time can be spent on showing us the techniques and strategies of solving problems on applications of the results ... ."

> "Another problem encountered is that graph theory questions are generally difficult to solve in the sense that there are very few standard questions. Every problem involves thinking as things are often not as simple as we thought and thus requires different strategies and approaches. Also, it is often difficult to present the solutions properly and systematically."

> "The workshops should address the issue of how to teach the subject ... so that we are better prepared to approach the students with some strategies to help them learn the higher level mathematics."

> "Students are too used to doing routine problems. It is also difficult for teachers to tell them that the reason why we can think of the solution is because of the experience we have in engaging in the topic."

It should not surprise us that the workshops have to be comprehensive to address the differing needs of the teachers. It is clear that content knowledge by itself is not sufficient. Just to name a few, the teachers need to know how to set out proofs, craft test questions, provide real life applications, and initiate students into the world of higher mathematics.

II Teachers' Beliefs about Discrete Mathematics

The teachers were asked to rate their agreement, from 1 (Strongly Disagree) to 5 (Strongly Agree), with nine statements on their beliefs about Combinatorics and the same nine statements about Graph Theory. In general, each teacher responded rather similarly to a given statement for both Combinatorics and Graph Theory. In what follows, we will use the mean of the responses to report the central tendency. Since the scores are not ratio data, the mean is not meant to be indicative of any 'real' value but only as a succinct indicator of the distribution of the responses. Means (μ) when reported will be first for Combinatorics and then for Graph Theory.

Almost all teachers strongly agreed (μ = 4.7 (Combinatorics), 4.3 (Graph Theory)) that solving problems in the topic require application of problem solving strategies. Diversity in methods (μ = 4.7, 4.0) and, to a lesser extent, creativity (μ = 4.1, 3.5) were perceived to be aspects needed for problem solving in Discrete Mathematics. On the other hand, teachers were neutral to the application of standard procedures (μ = 3.2, 3.2), and drill and practice (μ = 3.0, 2.9). This finding is in line with the teachers' comments on the need for future workshops to include showing them how to use problem solving strategies and how to teach them. This should help them deal with the challenges of teaching Discrete Mathematics, which they felt lacks standard problems and thereby call for some ingenuity and experience in solving problems. That "students are too used to doing routine problems", as one of teachers have remarked, makes teaching of Discrete

Mathematics more difficult. Such sentiments would support the finding that there is a slight slant towards believing that natural aptitude is what counts the most in Discrete Mathematics (μ = 3.8, 3.6).

The teachers were very positive about Discrete Mathematics, strongly endorsing its applications to real-life (μ = 4.4, 4.7) and perceiving it to be very interesting (μ = 4.7, 4.5). Comments from the teachers reported in Section I reveal that the teachers also felt that it can be motivating to the students if the lessons included real-life applications of appropriate mathematical demands. The challenge lies in identifying real-life applications which are sufficiently mathematically sophisticated but accessible to the students at this level.

III Teachers' Beliefs about Students Learning Discrete Mathematics at H3 Level

The teachers were asked to rate their agreement, from 1 (Strongly Disagree) to 5 (Strongly Agree), with ten statements on their beliefs about students learning Discrete Mathematics at H3 Level for Combinatorics and the same ten statements for Graph Theory. As before, in general, each teacher responded rather similarly to a given statement for both Combinatorics and Graph Theory.

The teachers had a balanced view of the teaching of procedures. They mildly disagreed with the recommendation that the teacher 'stress the correct procedures more than the problem solving strategies' (μ = 2.3, 2.9) and disagreed with the notion that drill-and-practice was *sufficient* for students to do well (μ = 2.0, 1.9). Of interest is their affirmation that drill-and-practice still has a part in learning Discrete Mathematics. The teachers agreed that drill-and-practice was *necessary* for students to do well (μ = 3.7, 3.9). (Trust us mathematicians to make a distinction between sufficiency and necessity!)

Showing a single correct solution was deemed insufficient for the teachers. Alternative solutions and why a solution works were aspects that they regarded very important. They agreed that teachers should discuss with the class different solutions to a problem (μ = 4.8, 4.8), and why a particular solution works (μ = 4.3, 4.1). They agreed as well that the focus must not be on the teachers only but that students should know how to verify their own solutions (μ = 4.3, 4.3).

This acknowledgement of student-centred learning revealed itself in their coolness towards the statement that teacher-centered lessons is the most effective way of teaching (μ = 2.7, 2.8) and their mild endorsement of student group discussion (μ = 3.6, 3.5), something rarely practiced in the exam-driven mathematics classroom of the junior college (from the authors' collective experience teaching in junior colleges).

The teachers showed a healthy respect for the ability of their students, agreeing that some students would be able to come up with solutions that the teacher had not thought of earlier (μ = 3.6, 3.7). They also acknowledged the need for time to solve problems in Discrete Mathematics by disagreeing with the assertion that students who take a long time to solve the problems are not capable (μ = 2.6, 2.5).

If we may take teachers' beliefs as strong influences on their choice of instructional approach and the way they engage the students in thinking mathematically, the preceding positive findings reassure us that the teaching of Discrete Mathematics may yet achieve the curricular aims as envisaged by its inclusion of the subject in the curriculum.

IV Problems/positives in teaching of Discrete Mathematics at H3 Level

The teachers were asked to rate their agreement, from 1 (Strongly Disagree) to 5 (Strongly Agree), with eleven statements on the problems/positives in teaching for Combinatorics and the same eleven statements for Graph Theory. This time, responses for Combinatorics and Graph Theory were sometimes different.

Five (62.5%) of the teachers were neutral to the statement that students were not ready for the advanced content of Combinatorics, with 2 (25%) disagreeing and 1 (12.5%) agreeing. On the other hand, the response for the similar Graph Theory statement was more varied and leaned more towards agreeing: Strongly Agree (12.5%), Agree (37.5%), Neutral (37.5%), Strongly Disagree (12.5%). Overall, we had for Combinatorics, $\mu = 2.9$, and Graph Theory, $\mu = 3.4$. The responses for the statement "Students are not able to appreciate the mathematical arguments and reasoning" were similarly bifurcated for Combinatorics ($\mu = 2.8$) and Graph Theory ($\mu = 3.3$).

The teachers were generally positive about the learning value of Combinatorics though less so for Graph Theory: The statement "The students mature mathematically in the course of this topic" had $\mu = 3.6$, 3.4 and the statement "There is a transfer of mathematical skills across to H2 Mathematics" had $\mu = 3.6$, 2.9. Almost all teachers (87.5%) responded that they would not feel threatened by students who were better than they were in the topic. When asked if they found it a joy to teach the topic, the responses were as follows: Combinatorics (SA – 1; A – 3; N – 1, D – 1; SD – 1) and Graph Theory (SA – 2; A – 3; N – 2). The last item in this paragraph taken with the earlier items was interesting because it seems to imply that Graph Theory when compared with Combinatorics was nicer to teach but had less 'value'.

On preparation, exercises and assessment, teacher responses expressed general difficulty:

- There is not enough suitable reference material for teachers. ($\mu = 3.3$, 3.5)
- It is hard to prepare tutorial questions. ($\mu = 3.4$, 4.0)
- It is hard to prepare test questions. ($\mu = 3.8$, 4.6)
- I am not sure to what rigour a solution requires. ($\mu = 3.8$, 3.6)

The teachers' written comments reported earlier agree with these findings. They suggest that future professional development workshops equip them with the know-how to craft questions for tests and examinations.

Teachers also felt that there was not enough time to cover the Graph Theory syllabus ($\mu = 3.9$) though they were neutral for the Combinatorics syllabus ($\mu = 2.9$). This finding appears to support the teachers' view that students are not ready for the advanced content of Graph Theory (they were split on whether the students are ready for the advanced

content of Combinatorics), and that Combinatorics has better "learning value" than Graph Theory.

V Teacher and Teacher-perceived Student Preferences for H3 mathematics topics

When asked to rate their own liking, from 1 (Dislike a Lot) to 5 (Like a Lot), the teachers responded very positively to both Combinatorics ($\mu = 4.5$) and Graph Theory ($\mu = 4.3$). They were modest in their self-evaluation when asked to rate their own proficiency, from 1 (Weak) through 3 (Neutral) to 5 (Very Good), responding with $\mu = 3.8$ for both Combinatorics and Graph Theory.

The teachers were asked to rank the four H3 mathematics topics in terms of their own liking, their perception of what their students like and the ease of teaching the topic. For the teachers, the overall ranking from top to bottom for their liking was Combinatorics, Graph Theory, Differential Equations and Plane Geometry. The ranking for teacher-perceived student preference was Differential Equations, Combinatorics, Graph Theory and Plane Geometry. The teacher-perceived student ranking did not match the teachers' own preference but on the other hand, it matched exactly the ranking of the ease in teaching of the topics.

These findings indicate that there is interest in Discrete Mathematics and a place for it in junior college mathematics. A likely explanation for the anomalous behaviour of H3 Differential Equations in the rankings of the survey is that students prefer topics which are comparatively more procedural and thus easier to do well in, and teachers also find them easier to teach.

Implications for Teaching of Discrete Mathematics in the Schools

This paper reports a first look into the teaching of Discrete Mathematics in Junior Colleges in Singapore from the perspective of a group of teachers. While the views of these people may not be representative because the sample is small and convenient, we can draw some implications and recommendations for the teaching Discrete Mathematics in the schools.

It is clear that teachers are looking for a blend of both content and pedagogy in training workshops to help them teach Discrete Mathematics better. The provision of content in the training workshop serves two groups of teachers. It is a refresher and update for teachers who have taken the relevant courses in Discrete Mathematics, and provides the content knowledge for teachers who have not. Among the pedagogical concerns revealed by the survey that should be addressed are the provision of group discussion techniques, skills to enhance the teacher's discussion of different approaches to the solution of problems, and ways of setting out proofs with their students.

The teachers also see it necessary to teach students problem solving strategies and techniques, as well as include real-life applications in their lessons. This would indeed support the argument for including Discrete Mathematics in post-secondary curriculum; the subject is accessible, provides for rigour in mathematical reasoning and writing, acts as training ground for mathematics and problem solving in context, and has various

applications. An immediate undertaking for us would be to meet the teachers' request for more real-life applications of Discrete Mathematics that are mathematically sophisticated but accessible to the students at this level. This support could be the development of curriculum materials (e.g., text books) or Internet resources.

Another immediate task for us is to address the twin challenges of equipping the teachers first with the problem solving strategies and then with the pedagogy of teaching problem solving to their students. What is the "best" way to these ends? For example, how best should students be introduced to problem solving strategies: Should problem solving be taught separately or embedded within the lessons? This task is all the more important because developing the students' ability to solve problems is the overarching aim of a mathematics education in Singapore schools. A step in this direction has been taken by a team of National Institute of Education researchers (Tay et al., 2007; Quek et al., 2007). They explored the notion of a 'Mathematics Practical' which they embedded within the framework of problem solving provided by George Polya (1954) and Alan Schoenfeld (1985). The Mathematics Practical is seen as a means of inculcating in students mental habits that enhance their problem solving performance. Emphasised strongly in the Mathematics Practical is the generation and formulation of problems based on the problem just solved. Thus, problem solving workshops for teachers in the line of the Mathematics Practical have this added benefit: the teachers in need of ways of crafting test and examination questions in Discrete Mathematics can be assisted as well.

The teachers' attempt in good faith to get their students to come to grips with the subject of Discrete Mathematics may result in them trying to "routinise" the problems; as one teacher puts it, "[W]e try to expose the students to as many questions as possible …." In this light, support for teachers in their teaching of Discrete Mathematics is crucial for students not to see Discrete Mathematics as another set of routines.

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