# The history of the teaching of the concept of a function in Russia

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We will briefly trace the history of the development of the concept of a function following the fundamental monograph edited by A.P. Yushkevich (1970, pp. 139-148, and 1972, pp. 250-255), and also books by F.Klein (1977, pp. 286-292), G.I.Gleizer (1983, pp. 20-25) and to the “Mathematical encyclopedic dictionary” (1988, p. 617). We also trace the history of the development and teaching of the concept of a function in 20-th century on the basis of the analysis of mathematical, didactical, educational and reference books and articles.

In the monograph edited by A.P.Yushkevich (1970) it is noted that “shoots of this concept though not yet realized and not allocated, were present in the Greek mathematics and natural sciences” (p.139). Searches of quantitative interrelations between various physical sizes in acoustics and astronomy were carried out in the Ancient Greece and Babylon.

“But the ancient science dealt not only with tabulated functions. In the theory of conic sections the basic role belonged to … the equations of these curves formulated verbally, and, unlike tables of values of strictly individualized functions, these equations express whole classes of correspondence, say, for coordinates of any ellipse, or a parabola or a hyperbole... The volume of the set of functions studied in the antique mathematics was not large but operations on them in a number of problems were pretty much similar to later ones: researchers studied their properties, tabulated, interpolated, found extrema, solved some problems equivalent to modern integration.

The technique of analytical expressions and symbolical formulas was absent …

The concept of a function for the first time emerged in the medieval Europe in connection with the renewal of attempts of mathematical studying of various natural phenomena …

For the subsequent development of the theory of functions crucial importance belongs to trigonometry and logarithms, on the one hand, and to the birth of symbolic algebra, on the other hand.

At the beginning of the 17-th century functions still quite often were defined verbally, graphically, cinematically or through tables, but already in the second half of the century their analytical expressions play the leading role.

The term "function" for the first time appeared in Leibniz’s manuscripts of 1673, in particular in the manuscript entitled “The inverse method of tangents, or on functions” (Methodus tangentium inversa, seu de functionibus). For the first time in a published form the word "function" was used in Leibniz’s article printed in Journal des Scavans in 1694. However, he gave then to the term much narrower sense, than the modern concept of function has.

In 1698, in the correspondence of Leibniz and his disciple I.Bernulli “… the word "function" was used already in the sense of analytical expression. The terms “variable” and “constant” were introduced by Leibniz” (ibid., p. 146).

The definition of function as analytical expression, for the first time was distinctly formulated by I.Bernulli's published in the article published in “Memoires de l'Academie des Sciences de Paris” in 1718. I.Bernulli wrote: “Definition. A function of a variable is a quantity composed in any way from this variable and constants”. He offered as a designation of a function the Greek letter fi… writing down the independent variable without brackets: $\varphi x$. Brackets, as well as the notation $f$ for a function, were introduced by L.Euler in 1734.

At the end of XVII-th century also examples of functions of two and greater number of variables appeared.

The further development of the concept of a function is connected first of all with L.Euler's name.

A.P. Yushkevich (1970) noted that “in the foreword to the “Introduction to the calculus of infinitesimals” he (L.Euler. – I.S.) for the first time has distinctly expressed the idea that the analysis is the general science about functions, that “the analysis of infinitesimals rotates around variable quantities and their functions (p. 250).

In the first chapter of the first volume of the “Introductions to the analysis of infinitesimals” L.Euler, correcting the definition of his teacher I.Bernulli, “…has emphasized that functions are defined by formulas: “A function of a variable quantity is the analytical expression composed somehow by its variable and numbers or constant quantities”. Thus the important step forward was done; the independent variable is considered as the set of all real and imaginary numbers so that functions of a complex variable were introduced on the equal rights with functions of a real variable” (p. 250).

By ways of composing analytical expressions were meant four elementary operations, computation of a root, exponential and logarithmic operations and, furthermore, “uncountable others, provided by integral calculus”, meaning thus also the integration of differential equations.

The Euler’s definition of a function given in the first volume of the “Introductions to the analysis of infinitesimals”, appeared to be too narrow for the calculus as a whole.

“…It has impelled Euler to give another, more general definition which, however, he had already used earlier. The concept of a function is meant as a correspondence between elements of sets of values of two variables, the concept having existed long since, but never before distinctly formulated as it was not required. In the first chapter of the “Introductions to the analysis of infinitesimals” Euler sometimes addresses to research of properties of functions whose analytical expression are not known in advance such as inverse functions, implicit functions or the function defined parametrically. Reasonings, by means of which Euler proves the existence of functions in these cases, are not strict at all, however, it is interesting, that functions in these cases are simply some correspondences. Similarly, in the second and third chapters it is explained that the same function can be presented by infinite number of various convertible to each other analytical expressions; the general substratum of all these expressions is some correspondence between elements of numerical sets.

Euler formulated the new definition of a function in his foreword to the “Differential calculus” (1755): “When quantities depend on others in such a way that at the change of the last they are also changed the first are called functions of the second ones. This term has extremely wide character; it covers all the ways by which one quantity can be determined by means of others”. In the quoted definition nothing is spoken about the way of calculation of values of a function.

Close to the modern definition of a function is N.I.Lobachevsky’s definition: “…the general concept of function demands, that one should name a function of $x$ a number that is given for every $x$ and that gradually changes together with $x$. A value of a function can be given either by analytical expression or by a condition which gives means to test all numbers and to choose one of them, or, at last, dependence may be unknown” (see the Mathematical encyclopedic dictionary, p. 617).

Thus, “… the classical definitions of a function given by N.I.Lobachevsky in 1834 and by L.Dirichlet in 1837 the second of which has passed to the latest textbooks (“If in some interval to each separate value $x$ a unique value of the variable $y$ corresponds then the variable $y$ is called a function of $x$”), are hereditarily connected with a definition belonging to Euler » (Yushkevich, 1972, p. 254).

F.Klein (1977, p. 291) noted that with the development of Cantor’s set theory “began to be considered also functions defined for values of $x$ from any set (not necessarily numerical)…”

In B.L.van der Warden's classical textbook in 1930 the, we already see the quite modern definition of a mapping: “If to each element from some set M by any rule a unique (generally speaking, new) object $\varphi(x)$ is put in a correspondence then this correspondence $\varphi$ is called a function. If all objects $\varphi(x)$ belong to some set N, the correspondence $x \mapsto \varphi(x)$ is called also a mapping from M into N » (van der Waerden, 1930).

F.Klein (1977, p. 292) complained that “…the school mostly ignores all development of a science which took place after Euler” and offered: “… we wish that the general concept of a function … has entered as the enzyme into all the teaching of mathematics at school; but it should be introduced not in the form of abstract definition but rather on concrete examples … in order to make this concept a living property of a pupil”. F.Klein noted that “it would be desirable that among numerous teachers there was at least a small number of independently working people who would be familiar also with the newest concepts of the theory of sets” (ibid.).

Despite the F.Klein's appeal, formulated by him (and attributed by him to Euler) and more precisely by van der Waerden definitions of a function did not soon find the path to the educational practice and literature not only in secondary but also in the higher school in our country. Up to the beginning of 21-th century only the first part of F.Klein’s appeal has been in essence executed: Euler’s definition of a function has taken a strong place in school and university mathematical curricula. The second part – taking into account the development of mathematics after L.Euler and use of set-theoretic concepts - actually is not executed till now at the school level and even not completely at undergraduate level.

So, in Soviet schools up to the middle of 60-th the textbook of algebra written by A.P.Kiselev prior to the October Revolution with the following definition of a function was used:

"That variable whose numerical values change depending on numerical values another one is called a dependent variable or a function of that other variable" (Kiselev, 1964, p. 25).

In the textbook it was spoken about tabulated and graphical presentations of functions; however the emphasis was made on the analytical expression of a functional dependence (in the form of the formula).

Similar definition was contained in the textbook of A.N.Barsukov for grades 6-8 used from 1956 to 1967:

"If two variables are connected in such a way that to each value of one of them a unique value of another one corresponds, one speaks that there is a functional dependence between these variables.

… If two variables are in a functional dependence, the variable that can accept any (admissible) values is called an independent variable...

Other variable, whose values depend on values of the former one, is called a dependent variable or a function…» (Barsukov, 1967, p. 250).

Still in 1970, in the algebra textbook of E.S.Kochetkova and E.S.Kochetkov for grade 10 that replaced A.Kiselev's textbook, the similar definition is given (in a little bit more precise form), essentially ascending to Euler's one:

"If to every value of a variable $x$ somehow a certain value of another variable $y$ is put in a correspondence one says that a function is defined» (E.S.Kochetkova and E.S.Kochetkov, p. 127).

Addressing to the foreign educational literature we see that in the published in 1941 classical book of R.Courant and H.Robbins "What is mathematics?" we again see essentially Euler's definition:
"…To each value of a variable X some certain value of another variable U is put in a correspondence. In this case a variable U is called a function of a variable X » (Courant, Robbins, 2000, p. 301).

The same situation is observed in undergraduate textbooks. So, in the textbook on higher algebra by G.M.Shapiro (1935, p.5) also the functional dependence is stressed:

"If two variables $x$ and $y$ are connected in such a way that to each value $x$ is a certain value of the variable $y$ corresponds, then the variable $y$ is a function of $x$: $y=f(x)$".

Similarly, in A.K.Sushkevich's textbook (1941) it is supposed by default that a function is an expression *f(x)*, where *x* is variable quantity (p. 86).

At the same time in A.Sushkevich's textbook there is (in a little bit archaic language) the quite modern definition of a group homomorphism with the requirement that to each element of the first group one has put in a correspondence a unique element of the second group, i.e. with the requirement to a homomorphism to be a mapping in the modern sense (p. 353).

Note that in the published much later, in 60-th, L.Okunev's textbook there is no concept of a group homomorphism, but only a concept of isomorphism. Among linear mappings only linear transformations of a vector space into itself are considered. At last, the same term "linear transformation" refers to both linear operators of a vector space and transformation of variables after the replacement of basis, and such use may cause confusion.

In A.G.Kurosh's higher algebra textbook used in 60-70-s, among group homomorphisms only surjective ones are considered, and among linear mappings of vector spaces only linear transformations of a space into itself are considered.

V.A.Uspensky (1965) noted that in the Great Soviet encyclopedia in 1956 a function was defined as a dependence of variables on other ones.

He also mentions similar definitions of a function in authoritative undergraduate textbooks on calculus:

"A quantity $y$ is called a function of a quantity $x$, defined on a set $M$, if to each value of a quantity $x$, defined on a set $M$, the unique value of a quantity $y$ corresponds" (Khinchin, 1953, p. 15); "A variable $y$ is called a function of a variable $x$ on the domain $X$ if by some rule or law to each value $x$ from $X$ a unique value $y$ is put in a correspondence" (Fichtenholz, 1964, p.40); "The law (rule) by which to values of independent variables the values of a considered dependent variable correspond, is called a function » (Myshkis, 1964, p. 37).

We explored a number of undergraduate textbooks on mathematics. Here are some results.

In the textbook of V.V.Stepanov (1953) on differential equations the definition of a function is absent at all.

On the other hand, in the second edition of the textbook of L.S.Pontryagin (1965) the special appendix was added containing the modern definition of a mapping (p. 292-293).

Note that the greatest mathematicians of the rank of N.N.Luzin, A.N.Kolmogorov, P.S.Aleksandrov, L.S.Pontryagin, apparently, were the first to realize the necessity of the introduction of the modern definition of a mapping into the scientific and educational literature.

Such definition of a mapping is used in books of N.N.Luzin (1948), P.S.Aleksandrov and A.N.Kolmogorov (1948), A.N.Kolmogorov and S.V.Fomin (1954). Note that all these descriptions characterizing a function as a rule of correspondence were not strict definitions and left the concept of a function (mapping) undefined.

At the same time some great scientists still did not introduce the general concept of a function and its definition, limiting themselves to special cases. So did A.I.Maltsev (1956) and I.M.Gelfand (1971) in their textbooks on linear algebra. Apparently, it was implicitly supposed, that mastering special cases of the concept of a mapping is enough for mastering the appropriate themes of mathematics, and it is not necessary "to multiply entities". The mathematics educator G.V.Dorofeev (1978, p. 21) expressed similar educational ideas when he in a discussion article even protected a thesis about uselessness of the definition of a function: "Pupils have, basically, the correct substantial view of a function as a mathematical object, but experience significant difficulties when they encounter the definition of this object… This situation, namely the possession of a concept without knowledge of its exact definition is not strange at all… it is typical in the majority of kinds of human activity…".

A.N.Kolmogorov (1978, p.29) in his reaction, however, indicated: "… G.V.Dorofeev … at school in general … allocates to any version of the set-theoretical definition of function a modest place (basically only for optional lessons). I think, however, that for school textbooks … rules (composing the definition of the concepts of a function. - I.S)… should be given to pupils early enough and should be coordinated with some certain final definition".

A.N.Kolmogorov supervised the reform of school mathematics teaching at the end of 60-s.

Meanwhile, in the textbook for upper secondary school edited by A.N.Kolmogorov a rather concrete definition of a function is used, and authors consider only numerical functions:

« A correspondence with a domain D where to each number $x$ from the set $D$ a unique number $y$ is corresponds by some law, is called a numerical function» (Kolmogorov et al., 1990, p. 20).

Thus, "Kolmogorov" reform did not aim at giving to teaching of mathematics abstract and formal character of what it was severely accused by opponents. The purposes were to eliminate archaic language and character of teaching, to correct the scientific level of mathematical education. The great attention was given to the didactical maintenance of the reform.

Further development of methods of introduction and teaching of the concept of a function became possible in 90-th years when it was allowed to use alternative textbooks at school. Some new complete sets of textbooks on school mathematics with rather successful approaches to the introduction of the concept of a function.

So, in the set of textbooks of A.G.Mordkovich for grades 7-11 the dialectic approach to the introduction of mathematical concepts is applied: "…the concept of a function … should not, in a deep belief of the author, be introduced strictly from the very beginning, it should "grow" (Mordkovich, 1998, p. 6). Strict definition of a function is introduced only in grade 9. Nevertheless, A.Mordkovich also considers only numerical functions, defining them in the language of variables.

We see a similar picture in the textbook of M.I.Bashmakov (1992, p.11):

"The variable *y* is a function of a variable x if such dependence between these variables is defined that allows for each value *x* uniquely determine the value of *y*".

Thus, the most widespread textbooks of mathematics for the senior grades contained definitions of functions which are similar to Euler's definition, mentioned by F.Klein. Has not remained without attention also F.Klein's advice to introduce these definitions gradually, on examples. We see, that the majority of mathematical and educational community admitted that it is inexpedient to study the modern set-theoretical definitions of a mapping at school.

Meanwhile, in the first post-war decades (1945-1960), in connection with fast development of topology and abstract algebra, in the world mathematics the most formal definition of a function (=mapping) as a such correspondence from one set into another (i.e. a subset of their direct product) where for any element *x* from the first set there will be a unique element *y* from the second set such that the pair (*x, y*) belongs to this correspondence.

Already in 60-eth, as many researchers observe, owing to "bourbakization" of mathematics, set-theoretical concepts and, in particular, the general concept of a mapping (function) began to enter into curricula of secondary and tertiary school.

In particular, the general definition of a mapping has been introduced into the first textbooks corresponding to reformed curricula and published under the edition of A.A.Markushevich (1975) - the prominent mathematician and educator, the ally of A.N.Kolmogorov in the reforming of school mathematics. In 1960-70-eth didacticians have developed also methods of the teaching the concept of a function at school (Kolyagin et al., 1977), and concluded that the concept of a function is expedient for studying at school quite strictly, consistently introducing concepts of the set, of the ordered pair, the direct product of sets and of correspondences.

Appropriate steps have been undertaken for the preparation of school teachers: in particular, into curricula of pedagogical institutes new subjects have been introduced: "Scientific foundations of school mathematics", "Modern foundations of school mathematics". Note that the introduction of these subjects has played a revolutionary role in reforming also undergraduate mathematical courses, having set new standards of strictness and modern mathematical language.

Nevertheless, in 80-s, after the publication of the notorious article of L.S.Pontryagin in the magazine "Communist", directed against Kolmogorov reforms, the general concept of a mapping, as well as other general set-theoretic concepts, has been expelled from school curricula, and Euler's definition of a function occupied a strong position in school mathematics.

In our opinion, it was a mistake, because the absence of the strict definition often complicates mastering the concept of a mapping by students in their further study in universities. We believe that, not demanding from pupils the faultless possession of strict definitions, it is necessary to attain nevertheless that they would be aware of the modern definition of function.

As it is already mentioned, in undergraduate textbooks, since 60-70-s and not without the influence of manuals on modern and scientific foundations of school mathematics, the strict definition of a mapping basically has gained a strong position: one may mention textbooks of L.A.Skornyakov and A.I.Kostrikin on algebra, of V.I.Arnold on differential equations, of V.A.Zorich on calculus. In some textbooks for pedagogical institutes (for example, in the textbook of L.Kulikov on algebra) the strictness and formalism have got excessive character, complicating the learning of a subject.

Nevertheless, in some undergraduate textbooks of the prominent scientists of the senior generation (for example, of D.K.Faddeev on algebra, of I.M.Gelfand on linear algebra) authors still tried to avoid the introduction of the general definition of a mapping.

Thus, the final opinion about the introduction of modern strict definition of a mapping is not reached concerning not only school, but even undergraduate textbooks.

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