MODELS OF STUDENTS' CONCEPTIONS ABOUT THE TANGENT LINE

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The results presented in this paper are a part of a doctoral study nearing completion which focuses on students' conceptions about tangent lines. The study explores the ways in which students who have studied tangents in school in different contexts (Euclidean Geometry, Analytic Geometry and Analysis): (i) perceive properties that are not generally valid as defining conditions of the concept and (ii) create new, often not-valid, properties out of the fusion of information from across the different contexts. Data was collected through a questionnaire administered to 182 first year mathematics undergraduates in Greece at the very beginning of their studies. Here I exemplify from the data analysis in the course of which several models of the tangent line have emerged based on the above properties.

INTRODUCTION AND THEORETICAL ASSUMPTIONS

Like most concepts in Analysis, the concept of tangent line at a point of a curve is a local notion (e.g. it is defined at a point as the limiting position of secant lines). However, in some cases, such as in the case of a tangent to a circle, it has some global characteristics: it keeps the circle in the same semi-plane, for example. Thus, studying the ways in which students think when they work with tangents is important, especially as they meet the concept in different contexts (e.g. Euclidean Geometry, Analytic Geometry and Analysis) that may require alternating between global and local perspectives. The significance of global or local perspectives of the tangent concept is particularly acute when the tangent line has more than one common point or when it coincides with the curve, as well as in the cases of inflection and edge points (Castela, 1995; Tall, 1987; Vinner, 1991).

One interesting aspect of the notion of tangent line is that, although the name (of the notion) remains the same, its definition varies across different curricular contexts. These different definitions give different meaning to the same name and this difference probably generates different concepts related to the same name. The reference to a notion through its name could entail the reference to one of these meanings namely one of the concepts that are connected with this name (Sierpinska, 1994). Sometimes students who have studied tangents in different contexts may perceive as defining conditions properties that are not generally valid. Some of these properties have been created implicitly by the adjustment of a property useful in one context as if it is adequate in another. In a earlier phase of the doctoral study this paper draws on (Biza et al., 2008), a study conducted with Year 12 students in Greek high schools who have met the tangent line in different contexts (Euclidean Geometry, Analytic Geometry and Analysis) suggested three different perspectives on the tangent: the *Geometrical Global* (global perspective on the tangent characterising the Euclidean Geometry context); the *Analytical Local* (local perspective on the tangent characterising the Analytical context); and the *Intermediate Local* (intermediate perspective on the tangent created by the application of the geometrical properties of circle tangent in a neighbourhood of the tangency point). Additionally, several *influential factors* on students' thinking about tangent lines emerged regarding specific geometrical properties of tangency. The five most prominent of these were as follows:

F1: The tangent line could have only one common point with the curve.

F2: The tangent line could have only one common point in a neighbourhood of the tangency point. This implies that the line may have another common point with the curve in the following ways: other tangency point visible on the graph (*F2A*); other common point not visible on the graph (*F2B*); other common point (tangency or not) visible on the graph (*F2C*).

F3: In any neighbourhood of the tangency point the tangent line could have an infinite number of common points with the curve. In this case the tangent line could coincide with the curve, or a part of it.

F4: There exists a tangent line at an inflection point. This aspect of the tangent line is crucial as the tangency at an inflection point entails that the line *cuts through* with the curve at this point and *splits* the curve in different semi-planes.

F5: There is no tangent line at an edge point in which the derivative from the left and the right are different real numbers.

Usually, information about one of the above *factors* is not enough to give us an image about the ways in which a student thinks about tangent lines. In the next phase of the study, that this paper focuses on, I investigated further how different combinations of students' choices regarding these *factors* form different models of students' conceptions about the tangent line and what information and imagery students evoke while they solve relevant problems. In the following, I present the study briefly; I describe the *models of tangent line* that emerged from the analysis; and, as an example, I describe and discuss some elements of students' justification concerning one of these models.

METHODS

Data reported in this paper was collected from a questionnaire administered to 182 first year university students (97 female) from mathematics departments in Greek universities. All participants had been taught about the tangent line in Euclidean Geometry, Analytic Geometry and elementary Analysis courses in Years 10, 11 and 12, respectively, but not yet at university as the questionnaire was administered at the beginning of their tertiary education.

The questionnaire included tasks in which the students had to describe in their own words the tangent line, question 1; to describe properties of it, question 2; to identify if a drawn line is a tangent line of a given curve, question 3 subdivided in 14 different curves; to construct the tangent line, if it exists, of a given curve through a specific point on the curve, question 4 subdivided in 15 different curves, or outside the curve, question 5; to provide definitions, to write the formula, and to apply formula in specific cases, questions 6, 7 and 8. In questions 3, 4 and 5 only the graph was provided and no formula of the corresponding equation was given. Students were asked to identify or construct the tangents based on the graphs and justify their choices. The proposed curves in questions 3 and 4 were chosen in order to integrate the five *influential factors* described above. For example, there existed cases in which the corresponding line: had more than one common point with the curve (tangency or not); coincided with the curve (or part of it); and, passed through an inflection point or an edge point.

Firstly, student choices in questions 3, 4, 5, 7 and 8 were characterised regarding their *correctness* and analysed quantitatively (Biza & Zachariades, 2006). Then, student responses to questions 1, 2 and 6 as well as their choices/justifications in questions 3, 4 and 5 were analysed qualitatively. As part of this analysis a coding was produced that described students’ choices with regard to the above *influential factors*, the context and the representations used in the responses and the consistency of student choices. Also for each student, a small narrative was produced describing the overall student profile as emerging from their response to the questionnaire (Miles & Huberman, 1994). Through the above analysis the students were classified in eight groups, each related to a *model of student conceptions of the tangent line*. These *models* are constructions related with the *influential factors* and integrate properties of tangent line (not necessarily mathematically correct) revealed by students responses. Some of these properties - most of them relating to the circle tangent - act as unconscious and *uncontrollable* images, in the term proposed by Aspinwall et al. (1997). Furthermore, within each model and through the justifications the students offer in their responses (e.g. in terms of representations, words and symbols they use) some potential *personal concept definitions* (with the meaning given to the term by Tall and Vinner, 1981) of the tangent line emerged.

RESULTS

Models of students conceptions about the tangent line

Through the analysis eight models of students’ conceptions about the tangent line emerged. These models are derived by different combination of choices regarding the first four *factors* (F1-F4). Some of these models are split in two sub models concerning the factor F5 (acceptance/sketch or not a tangent at an edge point). All the students, except 14, are classified in these eight models/sub models.

The Table below presents the eight models coded M1 to M8: the numbers in brackets in column one indicate the number of students classified to the corresponding model and “✓” or “×” indicates acceptance or rejection of the respecting *factor*. For some of these models, the existence of sub models and the related number of the students are indicated in the last column.

	F1	***F2A***	***F2B***	***F2C***	***F3***	***F4***	***F5***
***M1* (18)**	✓	✓	✓	✓	✓	✓	✓ (17) × (1)
***M2* (33)**	✓	✓	✓	✓	×	✓	✓
***M3* (13)**	✓	✓	✓	✓	✓	×	✓
***M4* (42)**	✓	✓	✓	✓	×	×	✓ (37) × (5)
***M5* (27)**	✓	✓	×	×	✓	×	✓ (24) × (3)
***M6* (15)**	✓	×	**?**	×	×	×	✓ (9) × (6)
***M7* (13)**	✓	✓	×	×	✓	✓	✓
***M8* (7)**	✓	×	×	×	×	✓	✓ (3) × (4)

Table: Models and influential factors

In the first model (*M1*) the tangent line could have more than one common point with the curve (tangency or not), there is a tangent line at inflection points and there is no tangent line at an edge point (with the exception of one student). In this model the coincidence of line and curve is acceptable. In the second model (*M2*) the tangent line could have more than one common point with the curve under the condition that there is a neighbourhood of the tangency point where this point is the only common point. There is a tangent line at inflection points and there is no tangent line at an edge point. In this model, contrary to the first one, the coincidence of line and curve is not acceptable. The third (*M3*) and the fourth (*M4*) models are similar to the first and the second, respectively, with exception in the case of inflection points where the tangent line is not acceptable. Additionally, more students of *M4* (5 over 42) comparing to the previous models accept the tangent at an edge point. In the fifth model (*M5*), the tangent line could have more than one common point with the curve under the condition that all of them are *tangency points*. Furthermore, there is no tangent line at an inflection point and for 24 of the 27 students of this model there is no tangent line at an edge point. In this model the tangent line could coincide with the curve as all the common points could be regarded as *tangency points*. In the sixth model (*M6*) the tangent line has only one common point with the curve, there is no tangent line at an inflection point and for 9 of the 15 students of this model there is no tangent line at an edge point. Obviously, according to this model the tangent line cannot coincide with the curve as it has more than one common point with the curve. However, sometimes students classified in this model accept or sketch a line, if the other common points are not visible/sketched (*F2B)*. In the seventh model (*M7*) the tangent line could have more than one common point with the curve under the condition that all of them are tangency points. There is a tangent line at an inflection point and there is no tangent line at an edge point. In this model, similarly to the *M5* the tangent line could coincide with the curve. Finally, in the eighth model (*M8*) the tangent line has only one common point with the curve with the only restriction this point to be the only common point between the curve and the tangent line. As a result the tangent line cannot have more than one common point (tangency or not); can pass through an inflection point; and, for 4 of the 7 students of this model, can pass through an edge point.

An example: Model M4

While the above description provides an overall picture of the commonalities within the student responses classified in the same group, within each group the students' justifications and their personal concept definitions vary considerably. What follows is a brief exemplification of this variation with regard to the student responses classified in model *M4*. In this model the main difficulties are observed with regard to tangency at an inflection point and the coincidence of the tangent and the curve.

A personal concept definition of the tangent line, connected with model *M4* was: *The tangent line of a curve at point A is a line passing through A with a slope equal to the derivative of the function at this point.* For example, a student wrote about the relation of line and curve that: "Point *A* is a common point of the line and the curve and if the curve has a function $f(x)$, then $f'(x_0)$, with $A(x_0,f(x_0))$ will be equal to the slope of the line". This *definition* agrees with the *formal definition* in the case of the graph of a differentiable function. But this student, when point *A* was an inflection point (Figure 2), responded: "If we consider the figure as two curves with *A* the only common point, then ε is the tangent of the curve. If we consider the figure as one curve, then ε is not the tangent of the curve and the point *A* is an inflection point". Probably the uncontrollable image of a tangent line that cannot *cut across the curve* is so strong that she concerned the inflection point as a point in which the function is not differentiable.

Another personal concept definition of the tangent line connected with *M4* was: *The tangent line of* *a curve is a line that has (at least) one common point with the curve, (has the same slope with the curve at this point) and does not 'cut across' the curve at this point.* In this case the emphasis is in the *cutting across* relation between curve and tangent line. For example, a student stated that the tangent line is "a line that has one or more common point with a curve without intersecting with the curve at point *A*". The use of the word *intersection* is not necessarily in accordance with its mathematical meaning (e.g. in mathematics the tangency point is one of the intersection points between the line and the function graph whereas in the case of circle when a line intersect the circle is not a tangent). The meaning that this student gave to *intersection* is derived through the use of it in his expressions and through his choices in the specific tasks of the questionnaire. For example, he accepted the cases of tangent with another common point (like the one in Figure 1) by writing "Line ε is a tangent line at the point *A* because it touches and does not intersect at this point with the function graph" and he rejected the line sketched at an inflection point (Figure 2) because "the curve is intersected by the line at the point *A* and is not touched by it".

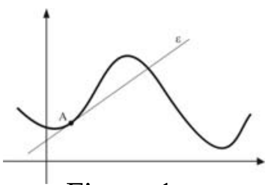

Figure 1

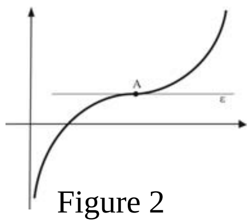

Figure 2

A third personal concept definition of the tangent line connected with *M4* was: *The tangent line of a curve is a line that has (at least) one common point with the curve, (has the same slope with the curve at this point) and there is a neighborhood of the tangency point in which the corresponding part of the curve remains in the same semi-plane of the line.* Although this statement was not expressed explicitly by students, it was detected through their reasoning and choices. For example, a student wrote in his description of the tangent line and its properties: "[…] all the points of the function [he means the function graph] are on the left or the right of the tangent line (at least the nearby points) […] if (ε): tangent and $f(x)$: function then $f(x)>(\varepsilon)$ or $f(x)<(\varepsilon)$ $\forall\ x \in [\alpha,\beta]$. It depends where the point *A* is". The same student accepted the tangent line even if it had another common point (like this in Figure 1) by writing: "It has only one common point and $f(x)>(\varepsilon)$". But he rejected erroneously the lines where point *A* was an inflection point or when the line coincided with the graph.

Finally let me refer briefly to the *local* perspective in the above description of the tangent line there is a neighbourhood around the tangency point, in which the curve remains in the same semi-plane of the tangent line. A similar statement of the remaining of the curve in the same semi-plane was applied *globally* by students, to the entire curve, in the *M6* model: *the tangent line of a curve is a line that has (exactly) one common point with the curve, (has the same slope with the curve at this point) and the entire curve remains at the same semi-plane of the line.* For these students the tangent line in both figures 1 and 2 should be rejected.

Another personal concept definition of the tangent line based on a *local* application of another geometrical property of tangent line (the property of the one common point) was: *The tangent line of a curve is a line that has only one common point with the curve in a neighborhood of the tangency point, (has the same slope with the curve at this point) and does not intersect the curve at this point.* For example, a student described the tangent line as a line that "has one common point with the curve in the interval (x-δ_1, x+δ_2), the point A". In accordance to this statement she could not accept as a tangent a line that coincides with the curve. Besides that, in the case of an inflection point, she rejected the line as "it intersects with the curve and it does not touch it" As in the previous case this statement is met in its *global* application: *the tangent line of a curve is a line that has only one common point with the entire curve, (has the same slope with the curve at this point)*, in models M6 and M8.

DISCUSSION

Through the above spectrum of models of the tangent line, we can observe that the circle tangent properties influence student choices although they are not always directly used. Students have transformed these properties into new ones. The *M1* could be concerned as the most adequate as students classified in it seem to have been released from the constraints of the Euclidean Geometry context. In the other models different kinds of mathematical (in)sufficiency in the student responses are revealed. I note however that overall there is no need to discuss these models in hierarchical terms as they do not represent each a more (or less) correct mathematical behaviour. Additionally this research aims to go beyond the observation of the ways that circle tangent properties influence student choices and brings out the ways in which the participating students evoke existing information and justify their choices about tangent while they solve relevant problems. What is significant about the models and the analysis of the students' justifications is that they reveal the different ways in which students apply terms, words, symbols, properties and representations met in a former context in new more general contexts as well as the ways in which this application takes place in a more (or less) effective fashion.

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