EXPLORING HIDDEN CONTEXT OF PRE-SERVICE TEACHERS' INTUITIVE IDEAS IN MATHEMATICS

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Paper prepared for TSG 29: The pre-service mathematics education of teachers
ICME 11, Mexico, 2008

Abstract

This paper suggests that mathematics teacher educators should listen carefully to what their students are saying. More specifically, the paper is written to demonstrate how from one primary pre-teacher's non-traditional geometric representation of a unit fraction, a variety of learning environments that lead to the enrichment of mathematical experiences for secondary teachers can be developed. The paper shows how new knowledge may be generated through an attempt to validate an intuitive idea; in other words, how the quest for rigor may serve as a catalyst for the growth of mathematical concepts in the context of K-16 mathematics.

Objective

The objective of this paper is to addresses two of the issues that are the main focus of TSG 29:

• How can one establish a link between the kind of mathematics and the role of mathematical experiences of the pre-service primary and secondary teachers?

• What are innovative approaches of developing mathematical content knowledge of pre-service mathematics teachers?

Towards this end, the paper shows how the mathematical content knowledge of pre-service secondary teachers can be developed in the context of ideas typically associated with the mathematics curriculum for primary pre-teachers. The paper also addresses the recommendations of the Conference Board of the Mathematical Sciences (2001) – an umbrella organization consisting of sixteen professional societies in the USA – for the preparation of school mathematics teachers. These include the need for courses connecting pre-college and college mathematics curricula. It has been a concern that prospective secondary school teachers do not see the connection between their undergraduate courses and the mathematics they are preparing to teach. To address this concern it has been recommended that an undergraduate capstone sequence be developed in which these connections are made explicit. By extension, such courses may also include examples of how undergraduate mathematics can be applied to even more foundational concepts that are the part of the primary mathematics curricula.

It has been widely observed that teachers of mathematics do not have sufficient mathematical background to see the general concepts behind particular phenomena. This lack of understanding contributes to the teaching of mathematics in a disconnected fashion, so that students in grade school believe that the problem solving work that they are doing is limited only to their grade level. Likewise, high school students are not able to see the connection between the problems they are currently engaged with and their earlier mathematical experience. To address this, teachers can use their knowledge of *hidden* concepts and structures of school mathematics to extend the curriculum in both directions.

Theoretical orientation

There are several related theoretical frameworks that underlie the ideas presented in this paper. One deals with the constructivist perspective on mathematical knowledge as a combination of action, operation, and reflection (Glasersfeld, 1995). Given this perspective, mathematics teaching is seen as a process in which both students and teachers learn. This leads to the emergence of a reciprocal learning relationship in which it is possible for teachers to learn from students as well. Steffe (1991) has introduced the notion of "possible learning environments" that mathematics educators can develop by going beyond the traditional expectations for students' learning and being willing to take the intellectual risk of including student-generated ideas in the design of instructional activities. In doing so, one can follow the constructivist perspective on mathematics education that privileges learners' ideas of mathematics over the conventional mathematics curriculum in making decisions about the kind of mathematics that should be taught (Steffe, 1991).

Another theoretical framework is based on the notion of hidden mathematics curriculum (Abramovich & Brouwer, 2003, 2006). This notion reflects the observation that many mathematical activities across the K-12 curriculum, while seemingly disconnected from a naïve perspective, are, in fact, permeated by a common mathematical concept or structure, traditionally (and, quite possibly, intentionally) hidden from learners because of its complexity. Such complexity may be either procedural or conceptual in nature. It should be noted that while the traditional conception of hidden curriculum (Shapiro, 2006) or, alternatively, hidden contract (Slater, 2003), has a negative connotation for learning, the notion of hidden mathematics curriculum as proposed is a positive learning mechanism. The authors' approach to investigating the idea of hidden mathematics curriculum in teacher education is to find, create, and work with problems across the curriculum that from a deeper perspective may be described by a common mathematical concept. Creating problems with, perhaps hidden, mathematical depth is consistent with Steen's (2004) call to help teachers to become better by providing them with rich problems that "convey the distinctive cohesiveness of mathematics" (p. 869). Technology has great potential to enhance this approach through appropriate pedagogical mediation.

Utilizing the hidden mathematics curriculum framework to design instructional activities provides a significant opportunity to enhance mathematics teacher education. The premise is that prospective K-12 teachers are given the opportunity to learn traditionally hidden, advanced mathematical ideas in the social context of competent guidance provided by university faculty. This social context may be supported and enhanced by the use of appropriately deployed technology tools. Embedding advanced ideas in the technology allows for their easier access. Such pedagogic mediation supports the advancement of Freudenthal's (1983) theoretical proposal of the didactical phenomenology of mathematics as a way to help teachers discover entries into mathematical culture developed over time. Put another way, technology-facilitated learning in the social context of expert-novice collaboration has the potential to uncover the hidden meanings of, what are commonly perceived as elementary, mathematical concepts.

A focus on pre-teachers' expertly facilitated mathematics learning is also consistent with one of the tenets of Vygotskian pedagogy in which the fundamental educative mechanism is interaction within a broader social context and learning is conceptualized as a transactional process of developing informed entrants into a culture with the assistance of more advanced participants, or "agents" of this culture (Bruner, 1985). Considering mathematical culture, it appears that this notion of hidden mathematics curriculum creates a powerful intellectual bridge between Freudenthal's didactical phenomenology of mathematics and Vygotsky's zone of proximal development (ZPD) that learning by transaction creates. The concept of the ZPD is based on the premise that human learning is, at its core, a social process in the sense that to fully characterize a learner's cognitive development, one must consider what the learner can accomplish with the support of a more expert facilitator. Vygotsky (1978) argued that learning by transaction creates the ZPD and thus "the only 'good learning' is that which is in advance of development" (p. 89).

At this confluence of pedagogical and psychological theories, the combination of Freudenthal's pedagogy of learning mathematics as the advancement of the culture of mankind and Vygotskian theory of learning in a social context provides theoretical underpinning for the pedagogical framework of hidden mathematics curriculum. In particular, teaching pre-teachers within such a framework in the social context of technology-enhanced learning creates the conditions for ZPDs to be developed – that in turn provide the basis for the pre-teachers' deeper understanding of fundamental mathematical concepts. It is through such expertly-scaffolded facilitation at the key points of the zone where assistance is needed (Tharp & Gallimore, 1988) that one develops the capacity and confidence to teach these concepts. Thus, a case can be made that a hidden mathematics curriculum framework enhanced by technology has potential to become a medium for what Vygotsky referred to as "good learning."

In what follows, the authors report on the development of teaching ideas that extend the notion of possible learning environments within the framework of hidden mathematics curriculum. It will be shown how possible learning environments emerge through uncovering hidden aspects of traditional mathematics curriculum. Hereafter, the term "possible learning environment" (or PLE) is used in this extended sense. Such an approach has the potential to expand teachers' knowledge base. Knowledgeable teachers, being in a better position to recognize mathematical meaning and potential in students' ideas, can make a real difference in the mathematics classroom (Lampert & Ball, 1998).

Methodology

Motivated by work done with elementary and secondary preservice and in-service teachers in various mathematics education courses as well as by field observation of school mathematical practice, this paper shows how technology tools, such as spreadsheets and dynamic geometry software, enable informal journeys into hidden aspects of the formal content of the school mathematics curriculum. It was found (Abramovich & Brouwer, 2007) that a hidden curriculum framework supports pedagogical mediation of formal mathematical reasoning by secondary pre-teachers, including the development of proof schemata and its analysis; the role of counterexamples; inequalities and their use as a vehicle for approximation; measurement as a means of developing special cases and generalization as a tool for understanding them; and the importance of historical perspectives. The appropriate use of technology is a thread that not only underlies all of these issues but, in addition, can motivate the development of new knowledge in mathematics.

The argument of the paper is supported by specific examples of student work that illustrate the above points. It is based on the analysis of classroom observations and portfolio assessments from a variety of mathematics education courses incorporating a technology-enhanced hidden mathematics curriculum framework. A few examples of possible learning environments emerging from one particular classroom interaction follow.

PLE 1: From square to pentagon

As mentioned elsewhere (Abramovich & Brouwer, 2007), in the context of a class discussion on geometrical representations of unit fractions, one elementary pre-teacher offered a representation of one-fourth through four self-embedded squares where the area of the smallest square is one-fourth of the area of the largest. It was shown how proceeding from this intuitive idea, a variety of possible learning environments can be constructed. The "one-fourth" idea can be extended to self-embedded regular pentagons through which different unit fractions can be demonstrated.

To this end, consider the sketch pictured in Figure 1, in which the area of the smallest pentagon is one-fifth of the area of the largest pentagon. Whereas this representation may appear intuitive at the elementary level, its formal demonstration requires the use of trigonometry – a major topic of the secondary mathematics curriculum. Indeed, one can show that the sides of the pentagons of areas 1, 2, 3, 4, and 5 square units have the lengths $a_1 = 2\sqrt{\frac{\tan 36^\circ}{5}}$, $a_2 = 2\sqrt{\frac{2\tan 36^\circ}{5}}$, $a_3 = 2\sqrt{\frac{3\tan 36^\circ}{5}}$, $a_4 = 4\sqrt{\frac{\tan 36^\circ}{5}}$, and $a_5 = 2\sqrt{\tan 36^\circ}$ linear units, respectively. Therefore, the areas of triangles with the sides of length a_1 and a_5 are in the ratio of 1 to 5. In general, one can use this approach to represent 1/*n* through *n* self-embedded pentagons with the sides $a_n = 2\sqrt{\frac{n \tan 36^\circ}{5}}$.

PLE 2: Embodying a mathematical definition in software

Geometric construction can be done with the help of the dynamic geometry software *The Geometer's Sketchpad* (*GSP*). To this end, one can construct a pentagon of radius $R_1 = \sqrt{\frac{2}{5\sin 72^\circ}}$, then define $R_2 = \sqrt{R_1^2 + \frac{2}{5\sin 72^\circ}}$ (in general, $R_{i+1} = \sqrt{R_i^2 + \frac{2}{5\sin 72^\circ}}$), and iterating this process *n*-1 times, obtain a sequence of *n* self-embedded pentagons. Such a sequence of pentagons can be dilated with the coefficient k. In this case, $R_1 = \sqrt{\frac{2k^2}{5\sin 72^\circ}}$ and $R_{i+1} = \sqrt{R_i^2 + \frac{2k^2}{5\sin 72^\circ}}$. Figure 1 shows the case for $k = 3$, $n = 5$. Learning to use *GSP* as an iterative tool illustrates "the way in which software can embody a mathematical definition" (Conference Board of the Mathematical Sciences, 2001, p. 132) as it involves the development of the skill of representing functions through a recursive definition.

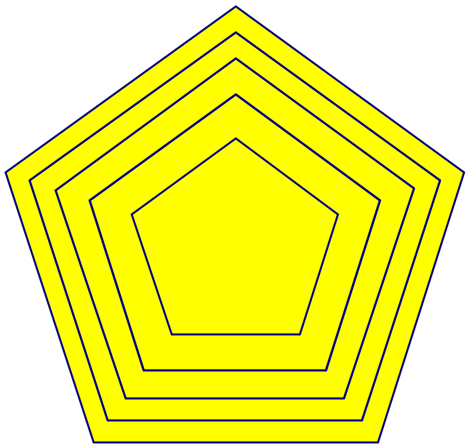

Figure 1. Iterated pentagons.

PLE 3: Discovering the Golden Ratio through measurement

If one measures the ratio of a diagonal to a side across all regular pentagons, this ratio appears to be the same regardless of the size of a pentagon. The ratio is approximately equal to 1.61803. Recall that two quantities *a* and *b* are in the Golden Ratio if the ratio between the sum of these quantities and the larger one is the same as the ratio between the larger one and the smaller one. By substituting f for $\frac{b}{a}$ in the equation $\frac{a+b}{b} = \frac{b}{a}$ one gets $1 + \frac{1}{f} = f$ or $f^2 - f - 1 = 0$ whence $f = \frac{1+\sqrt{5}}{2} \approx 1.61803$. In that way, one can discover the Golden Ratio in a regular pentagon. How can this be explained?

PLE 4: Complex numbers as a tool in making connections

Complex numbers can be used to explain the emergence of the Golden Ratio in a regular pentagon. To this end, let *d* and *a* be a diagonal and side of a regular pentagon, respectively. One can show that $\frac{d}{a} = \frac{1}{2\cos\frac{2p}{5}}$. According to de Moivre's formula $(\cos\frac{2p}{5} + i\sin\frac{2p}{5})^5 = \cos 2p + i\sin 2p = 1$. Therefore, $l = \cos\frac{2p}{5} + i\sin\frac{2p}{5}$ satisfies the equation $l^5 - 1 = 0$, which is equivalent to $l^4 + l^3 + l^2 + l + 1 = 0$. Dividing the last equation by l^2 yields $l^2 + l + 1 + \frac{1}{l} + \frac{1}{l^2} = 0$ whence $(l + \frac{1}{l})^2 + l + \frac{1}{l} - 1 = 0$. Applying de Moivre's formula yields $l + \frac{1}{l} = \cos\frac{2p}{5} + i\sin\frac{2p}{5} + \cos\frac{2p}{5} - i\sin\frac{2p}{5} = 2\cos\frac{2p}{5} > 0$. On the other hand, solving the last quadratic equation results in the equality $l + \frac{1}{l} = \frac{-1+\sqrt{5}}{2}$. Finally, the ratio $\frac{d}{a} = \frac{1}{2\cos\frac{2p}{5}} = \frac{1}{l + \frac{1}{l}} = \frac{1+\sqrt{5}}{2} = f \approx 1.61803$. Hence, one can use complex numbers to uncover hidden meaning of the presence of the Golden Ratio in a regular pentagon.

PLE 5: From fractions to calculus

Note that whereas the difference between the areas of two consecutive pentagons remains constant, as the pentagons grow larger, the difference between their perimeters is a variable quantity. One can use a spreadsheet, software commonly referred to as "one of the most widely used computer tools" (Conference Board of the Mathematical Sciences, 2001, p. 127), to explore numerically the behavior of the function $f(n) = \sqrt{n} - \sqrt{n-1}$ that represents this difference to the accuracy of a constant factor ($2\sqrt{\frac{\tan 36^\circ}{5}}$ in the case of pentagons). In doing so, one can see that the function $f(n)$ decreases monotonically and becomes smaller and smaller as n increases. Therefore, technology can be used as motivation for a formal demonstration of this phenomenon; indeed, "the feeling for rigor can be much better learned from examples" (Ahlfors, 1962, p. 190), examples that nowadays can be computer-generated.

Spreadsheet modeling can then be followed by the use of traditional techniques of the theory of limits studied in calculus. To this end, one can show that

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} (\sqrt{n} - \sqrt{n-1}) = \lim_{n\to\infty} \frac{n-(n-1)}{\sqrt{n} + \sqrt{n-1}} = 0.$$

This analytical property of perimeters of self-embedded pentagons can be given an alternative interpretation in terms of the following arithmetical property of natural numbers: whereas the difference between any two consecutive natural numbers remains equal to one as the numbers grow larger, the difference between their square roots varies and tends to zero. This abstract statement when situated in a geometric context can really come alive for prospective teachers.

While invoking trigonometry does foster the problem-solving skills of the teachers, it does not bring about new difficulties in terms of the theory of limits – the same result is true for any two consecutive regular m-gons. Indeed,

$$\lim_{n\to\infty}\left(\sqrt{\frac{n}{m}\tan(\frac{p}{m})}-\sqrt{\frac{n-1}{m}\tan(\frac{p}{m})}\right)=\sqrt{\frac{1}{m}\tan(\frac{p}{m})}\lim_{n\to\infty}(\sqrt{n}-\sqrt{n-1})=0.$$

In that way, one can "revisit the elementary functions of high school mathematics from an advanced standpoint" (Conference Board of the Mathematical Sciences, 2001, p. 43) and, in doing so, establish explicit connections between pre-college and college mathematics curricula.

Conclusion

The primary message of this paper is that in educational contexts mathematics educators should listen carefully to what prospective teachers are saying. In doing so, it is possible not only to respond in a careful and considered way, but also, as this paper intends to illustrate, through reflection, to develop learning environments that help students make meaningful connections across the K-16 mathematics curriculum. Furthermore, such environments may lead to the enrichment of mathematics for teaching. For instance, as mentioned elsewhere (Abramovich & Brouwer, 2007), by pursuing a naïve student idea of representing one-fourth, one can encounter non-trivial inequalities thereby developing new mathematical knowledge in an educational context.

Mathematics content enriched in this way, when presented as part of the capstone sequence, serves another purpose. Indeed, it provides a model of inquiry that can influence prospective mathematics teachers' conceptions of how new knowledge may be generated through an attempt to validate an intuitive idea. In such an educational setting, one can experience that just as the complexity of a symphony stems gradually from an unpretentious tune, mathematical concepts may grow from using intuition as a springboard for rigorous inquiry. To put it differently, it is the quest for rigor that serves as a catalyst for the growth of concepts.

Cognitive diversity, a pillar of constructivism, is commonly seen as a vehicle for inclusion. This inclusion may be interpreted broadly to embrace all participants of the learning community, in particular students with natural drive for curiosity. By drawing on the remarks of such students, not only can their curiosity be reinforced, but the curiosity of others can be motivated. These others may include teachers. Curious teachers are in a better position to follow the maxim: *Each day, try to teach something that you did not know the day before.* Such understanding of diversity could benefit courses in mathematics for prospective teachers.

In conclusion, the authors argue that through exploring possible learning environments designed within the framework of hidden mathematics curriculum, one can come across simple relationships associated with rather sophisticated contexts. This contextual complexity of primary mathematics concepts is a result of taking the constructivist focus on mathematics pedagogy seriously. Such an approach has the potential to uncover the hidden context of K-16 mathematics curricula and, furthermore, inform the capstone sequence for prospective secondary school teachers.

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