The mathematical preparation of secondary mathematics schoolteachers: Critiques, difficulties and future directions

Jérôme Proulx, *University of Ottawa, Canada*
Nadine Bednarz, *Université du Québec à Montréal, Canada*

First author contact information:
Faculty of Education, University of Ottawa
145 Jean-Jacques Lussier
Ottawa, Ontario
K1N 6N5 Canada
Tél. / Ph.: 613.562.5800 ext.4659
Télécopie / Fax: 613.564.5146
e-mail: jerome.proulx@uottawa.ca

Abstract: This paper offers a theoretical discussion of the potential orientation for the mathematical experiences and education of pre-service secondary teachers. Through an overview of the current literature on secondary mathematics teachers and the nature of the mathematical experiences they receive through teacher education practices, we outline and suggest specific points that appear relevant for the mathematical education of future mathematics teachers. Specifically, issues of school mathematics, in opposition to academic mathematics, and of engaging teachers in a culture and practice of mathematics, in opposition to receiving accepted knowledge, are highlighted and argued to represent fundamental dimensions that need to be present in our teacher education initiatives in regard to the mathematical education of teachers at the pre-service level.

> I am never critical toward teaching as it is practiced. If you see 200 000 teachers doing the same thing and that it looks stupid to you, it is not because there are 200 000 stupid people. It is because there is a phenomenon that orients this same type of reaction in these people. And it is this phenomenon that we need to understand. […] We won't improve it with an ideology, nor by moralizing to teachers.
>
> – Brousseau, 1998, our translation

Introduction

This paper has a theoretical focus as it reports, through a review of the current literature on secondary mathematics schoolteachers' knowledge of mathematics, on potential and fruitful directions for the mathematical education of mathematics teachers; something that is directly in line with the main focus and scope of this topic study group (TSG 29). The intention of this theoretical reflection is to stimulate exchanges and discussions for future directions and renewed practices of teacher education. We first report on current critiques addressed toward the mathematical preparation of secondary schoolteachers and follow this by reporting on research about secondary mathematics teachers' knowledge of mathematics. This leads us to outline one of our recent research projects focused on teachers' development of mathematical knowledge and draw orientations from it for teacher education practices. All of these issues are then used to orient and outline potential future directions for teacher education practices and research.

Current context concerning the mathematical preparation of secondary teachers

The literature on teacher education identifies recurring and important issues of disconnection between teacher education and teaching practices in schools. Currently in Canada, as well as other countries, secondary teachers are commonly required to take an important number of courses in subject specialisation, like mathematics, in order to teach at the secondary level. This issue has received recent attention particularly in regard to the gap created between the mathematical experiences teachers encounter in their university courses and their professional teaching practices in schools.

In effect, mathematical content knowledge, as a fundamental component in most mathematics teacher education programs, is usually identified with academic mathematics (which often are courses offered for future mathematicians). This subject-matter preparation for mathematics schoolteachers is often conceived as a self-contained process, implicitly promoting values, techniques, methods, conceptions and ways of thinking proper to academic mathematics, but not necessarily proper to the practice of teaching mathematics in schools. This orientation has significant implications for the constitution of schoolteachers' professional ways of knowing and practicing mathematics. Hence, while these courses are important for future mathematicians, it is questionable whether they are of value to mathematics teachers. In the following, we outline some of the concerns that have been raised in the research literature.

Mathematics teacher education: Issues about mathematics content, form and practices

Researchers have articulated the concern that the advanced and formal nature of the mathematics worked on in most of these academic mathematics courses could have the detrimental effect of reinforcing the abstract, technical and formal side of mathematics in teachers' understanding of concepts as well as in their teaching (Ball, Lubienski & Mewborn, 2001; Cooney & Wiegel, 2003; Gattuso, 2000), leading to serious difficulties in teachers' ways of making mathematics comprehensible and accessible to students.

Thomspon and Thompson's (1994, 1996) study of Bill, a mathematics teacher, is an illustration of these difficulties. Bill had a robust understanding of rate and speed at a formal and higher mathematical level, but this knowledge was so tightly woven and hidden under calculations and operations that it made him unable to articulate clearly these understandings in order to make the notions accessible and conceptually sound for his student. Bill's formal knowledge led him to perceive the connections and meanings as obvious, where he consistently used operations and calculations that *for him* made the connections explicit but left them opaque for his student, creating a gap between them. A second study comes from Nathan and Koedinger (2000) who administered questionnaires to teachers requiring them to rate a list of algebraic problems by order of predicted difficulty for their students. They found that the correlation between the lived difficulties of students and the teachers' predictions was very low, where teachers had overestimated the facility that students would have with formalism and symbolic manipulations. The researchers conjectured that teachers' own facility with symbolic manipulations led them to undervalue the difficulties that students could experience with these abstract forms. Discussing the study, the NRC (2001) expressed that university-level mathematics content knowledge "by itself may be detrimental to good teaching" (p. 399).

In addition to the formal aspects of academic mathematics, the nature of the concepts taught appears at odds with what teachers do when they teach, augmenting the disconnection toward their teaching practices. One characteristic and strength of academic mathematics is to make mathematical understandings, concepts and ideas "compact" and "compressed" so that they are more efficient, powerful, and easier to utilize (Adler & Davis, 2006; Ball & Bass, 2003; Moreira & David, 2005). Yet, as these researchers mention, it is the opposite that appears relevant and suitable for the teaching of mathematics to students. Teachers have to be able to untangle, unpack, dismantle, and decompress mathematical concepts to unearth the meanings, relations and subtleties hidden within their compact structure in order to foster and promote students' mathematical understandings. Mathematics teaching practices, as studies have shown, require a return to the concepts' elementary and underlying meanings in order to promote robust mathematical comprehensions in students (Bednarz, 2001; Brousseau, 1998; Ma, 1999); emphasis on the concepts' elementary and underlying meanings being outside of academic mathematics' focus. These differences are well illustrated by the analysis of the same content for both settings. For example, "According to academic mathematics, a rational number is just an equivalent class of ordered pairs of integers under the relation (a,b)~(c,d) ? ad=bc" (Moreira & David, 2005, p. 3). However in schools, the work on rational numbers relies on extending the number system from natural to rational numbers and, in introducing rational numbers, a "teacher has to consider that, so far, the child can only recognize the positive integers as numbers; in helping students develop the idea of rational number, a crucial step is extensive work with the various meanings of the fractions, as well as with other forms of interpreting a ratio of integers" (p. 3). For the same mathematical content, *named identically*, very different and disconnected approaches are used. Therefore, through its insistence on formalism and abstraction, the work in academic mathematics does not focus on developing the knowledge teachers will use in their professional teaching practices.

Another important issue about having teachers take academic mathematics courses concerns the way in which these courses are taught. As Bauersfeld (1998) and Burton (2004) explain, the usual way academic mathematics courses are taught is through modes of lecturing and the exposition of mathematical knowledge[1]. The ways of doing and habits developed in these courses are therefore more about "standardized knowledge" than about a participation in a *process of learning* that reflects teachers' practices. For Bauersfeld, teachers need to be immersed in a practice of doing mathematics, a culture of mathematics, rather than introducing them to a body of objective knowledge (where mathematics is an epistemological absolute). Participating in a culture of mathematics is to enter into a practice that uses mathematics, that shares and negotiates its meaning, that generates ideas, questions, norms and ways of doing in mathematics; a practice in which mathematics is created and alive.

These issues about the inadequacy of academic mathematics content, form and practices for schoolteachers, augmented by a research literature that fails to demonstrate a significant relationship between the number of academic mathematics courses taken by a teacher and students' performances (Begle, 1979; Monk, 1994), reinforce Bauersfeld's argument that mathematics teacher education models simply overestimate the positive effects of the academic training in mathematics on schoolteachers. This situation points to the inadequacy of identifying academic mathematics as representing teachers' subject matter preparation in teacher education programs, and it prompts reflections toward other sorts of mathematical experiences that could be offered to teachers.

Research on teachers' knowledge of *school* mathematics

Usiskin (2001) explains that when secondary mathematics teachers enter teacher education courses, they have not worked on school mathematics concepts since high school and have therefore important gaps in their knowledge of school mathematics and appear no "better prepared in the content they will teach than when they were students taking that content" (p. 2). There is indeed some research that points to secondary teachers' difficulties with aspects and concepts of the mathematics they teach. For example, studies by Ball (1990) and Bryan (1999) have illustrated that while the secondary mathematics teachers they studied made few if any mistakes in their usage of mathematical procedures, they experienced significant difficulties in providing sound meaning and explanations for the mathematical rationales underlying these same procedures. Quoting from Mewborn (2003): "By and large, teachers have a strong command of the procedural knowledge of mathematics, but they lack a conceptual understanding of the ideas that underpin the procedures" (p. 47). Other studies have highlighted difficulties of a different order concerning secondary teachers' unfamiliarity with the meaning of concepts themselves and solving processes (e.g., definitions, conjectures, relationships within concepts, etc.). For example, Even (1993) and Hitt-Espinosa (1998) have observed that many teachers possessed an "old" definition of a function as a continuous graph, preventing them from recognizing or accepting alternative drawings as representing a function and leading them to transform or treat discrete functions as continuous. Also, Schmidt and Bednarz (1997) and Van Dooren, Verschaffel and Onghena (2003) have reported on secondary teachers' difficulties in appreciating arithmetical procedures as valid solutions to traditional algebraic problems.

While these types of studies have been criticized as presenting a "deficit model" of teacher knowledge, and moreover cannot necessarily be generalized to all teachers, they nevertheless offer significant and insightful information. Thus, the issue is not about what teachers know or do not know, but about how we inform ourselves and learn from these studies. In other words, "What do these findings tell us about what we could or should offer teachers as mathematical experiences?" These studies show that our teacher

[1] However, it is not to say that "all" academic mathematics courses are taught in that way, as innovative approaches have been documented in the literature (e.g., Schoenfeld, 1985; Jones, 1977).

education practices do not succeed in enhancing teachers' knowledge of school mathematics, and they therefore put forth the need to provide teachers, through our teacher education practices, with opportunities to study and explore school mathematics concepts as well as enrich their knowledge of it.

Both sets of issues highlighted in the studies in the above sections are of concern. First, studies are underscoring negative effects of the academic mathematics education on secondary mathematics teachers and how it disconnects them from school mathematics. Second, and in addition, how our teacher education practices fail to address teachers' knowledge of school mathematics. This raises, in our view, the importance of addressing issues of school mathematics through teacher education practices to reconnect teachers with this mathematics taught in schools.

A teacher education project focused on the exploration of school mathematics

In such a perspective of offering opportunities to teachers to study and explore school mathematics concepts, a professional development (PD) initiative focusing on secondary mathematics teachers' exploration of school mathematics content was recently undertaken in one of our research studies (Proulx, 2007a). Working with teachers that had strong procedural skills, the intention was to enhance their knowledge through the exploration of school mathematics concepts. A primary goal of the PD was to have teachers deepen their understanding of the mathematics they teach everyday.

The research focused on the sort of learning opportunities that emerged from having the mathematics teachers explore and work on school mathematics concepts. The research's main findings concerned how teachers, through investment in diverse mathematical tasks and materials, enhanced their mathematical understandings and developed new comprehensions of the very concepts they teach everyday. Simply put, teachers learned a lot of/about the mathematics (that they teach). Teachers expressed how they gained access to novel aspects of school mathematics and engaged with these concepts concerning their meaning in ways they to never had the chance to do before in their education and were unfamiliar to them; it opened new ways of making sense of the mathematical concepts for the teachers. Through a focus on developing mathematical meaning and displaying reasoning behind the concepts (e.g., the meaning behind a specific mathematical procedure or formula, the signification given to particular symbolism, the relationships existing between concepts), these renewed mathematical comprehensions appeared to open new possibilities for approaching the concepts in teaching, as it led teachers to engage in profound pedagogical/didactical reflections and discussions concerning the teaching of these concepts. By enlarging their understanding of the mathematical concepts, teachers also enlarged their understanding of what teaching this mathematics implied. For example, in addition to discussions about student learning and teaching strategies, teachers described how their newly developed mathematical understandings were drastically changing their views about possibilities for teaching these concepts, often saying that they could not teach these concepts in the same way they used to because these were now much richer for them. In sum, though these professional development sessions focused on developing mathematical understandings of school mathematics concepts, teachers developed simultaneously what Cooney (1994) calls *mathematical* and *pedagogical* powers. These practices of teacher education appear of relevance because this sort of work offered teachers new possibilities for mathematics and its teaching that they simply did not have before.

The research also points to and reminds us of fundamental aspects concerning secondary-level mathematics teachers. First, even if secondary mathematics teachers may need to enhance their knowledge of school mathematics, they nevertheless possess important knowledge of mathematics; even if some of this knowledge is reported to be quite technical or procedural (one is reminded that knowledge of procedures is of great importance in mathematics). This knowledge needs to be built on and be used as a springboard to enrich, enhance, re-elaborate and deepen teachers' mathematical knowledge. Second, secondary teachers demonstrate a strong interest in knowing mathematics and are very curious to learn more. They have had, for the most part, a lot of success in school mathematics as students and they have indeed chosen a career in teaching it. Their professional identity is strongly related to mathematics

(Bednarz, Baribeau, Blouin, Gattuso, Lebrun & Lebuis, 1999[2]) and, as Cooney and Wiegel (2003) explain, they welcome and "enter" well into its study, making an entry through "mathematics" a fruitful way to engage secondary teachers in professional development. Third, even if obvious, one is led to realize that secondary mathematics teachers *can* learn a lot of mathematics. Hence, it is not about what they know or do not know, but about the fact that their orientation, appreciation and knowledge and understandings of mathematics leads them to be able to know and learn more and make new sense of mathematical concepts. Fourth, work on exploring school mathematics appears to support and provide reflections about the teaching of this mathematics. Even though this particular research project was completed with experienced teachers having a significant teaching background, student teachers still have teaching knowledge gained through their many years of schooling, knowledge on which they can build. As well, the explorations and learning teachers gain at the mathematical level can lead to significant reflections about how one can approach these mathematical concepts in teaching. An entry through mathematics is not restricted to mathematics itself but opens to pedagogical reflections for the teaching of this mathematics. There appears consequently to be a need for us, as mathematics teacher educators, to seriously realize, harness and take advantage of this context in order to participate in the continuous growth of mathematics teachers mathematically, and pedagogically.

Future and potential orientations for the mathematical preparation of secondary schoolteachers

The studies cited above raise important questions about the current orientations of mathematics teacher education programs and suggest a rethinking of the sorts of mathematical experiences and learning opportunities offered to teachers. We see two major themes arising from this discussion in relation to the nature of the *mathematical* experiences and knowledge teachers should be engaged in. We elaborate on these below, and offer some possible examples of related tasks.

A focus on school mathematics

One thing the above-cited studies underline is that teachers mathematical difficulties do not concern university-level academic mathematics courses, but rather the school mathematics concepts that teachers teach in their everyday practices. The disciplinary mathematical content worked on in teacher education needs to be the school mathematics concepts that they will teach in their everyday practices (e.g., algebra, fractions, analytic geometry, volume of solids, area of planar figures, etc.). Teachers need to be offered tasks and situations involving school mathematics concepts in order for them to enlarge and enhance their knowledge of the school mathematics terrain – knowledge directly relevant to their professional practices in school. And, above possible difficulties they can have with the concepts, as Ball (2003) explains, teachers are from the very mathematics teaching system we are trying to improve and render richer. There is therefore an incredible opportunity in our teacher education programs to offer mathematics environments that teachers may not have experienced as students.

However, we mean more by "school mathematics" than the mathematics prescribed by the curriculum. The sort of mathematical experiences we are talking about are oriented by Moreira and David's (2005) theoretical distinction that academic mathematics and school mathematics are different fields of knowledge. They use the term *academic mathematics* to refer "to the scientific body of knowledge produced by the community of professional mathematicians," whereas *school mathematics* is defined as "the set of validated knowledge, specifically associated with the development of school education in mathematics […] includ[ing] knowledge produced by mathematics teachers in their school practices […] as well as knowledge produced by research on teaching and learning of mathematical concepts and processes at school" (pp. 1-2). School mathematics represents not only the notions and concepts present in curricular documents and materials that outline what teachers have to teach, but the mathematical elements that surround them and that emerge in its learning and teaching. For example, in the teaching and

[2] Among other things, this study puts forth some issues of difference in relationship with mathematics of elementary and secondary teachers. It illustrates how prospective secondary teachers associated themselves strongly with mathematics, seeing it as an important part of their identity. This was not the case for elementary teachers.

learning of mathematics concepts, various related *mathematical* issues unfold: key reasoning, specific approaches and ways of making sense of concepts, range of specific procedures, strategies and representations, different misconceptions or difficulties experienced, and even aspects of the historical evolution of a mathematical concept. For teacher education practices, theorizing school mathematics in that sense "move[s us] away from the idea of school mathematics as a discipline taught at school to re-conceptualize it as a body of knowledge specifically associated with mathematics teaching at school" (p. 2). It is this body of knowledge that we deem significant for the mathematical education of pre-service teachers.

As a way of illustration, here are some examples of possible tasks related to this work on school mathematics, taken from our own work with prospective and practicing teachers. A first example is taken from Proulx (2007a, 2007b) about an algorithm to divide fractions:

> A colleague of mine coming back from China reported to me this procedure to divide fractions used by an 11 year old:
>
> $$\frac{26}{20} \div \frac{2}{5} = \frac{26 \div 2}{20 \div 5} = \frac{13}{4}$$
>
> Is this procedure correct?
> Does it work all the time? How?

Another example comes from Bednarz, Gattuso and Mary (1995) on comparison of decimal numbers:

> You are a 7^{th} grade teacher. You offer this exercise to your students:
>
> > Write the following numbers from the smallest to the biggest:
> > 2.46 2.254 2.3 2.052 2.32
> > Many of your students have written:
> > 2.052 2.3 2.32 2.46 2.254
> > And other wrote:
> > 2.052 2.254 2.32 2.46 2.3
>
> Complete the following steps:
>
> - Answer to the ordering of decimal numbers yourself;
> - Describe the error or errors committed by students;
> - Write a similar problem in which the students' reasoning would lead to the same mistake;
> - Write a similar problem in which the students' reasoning would lead to a right answer;
> - How would you intervene on these mistakes?

A focus on the mathematical culture

In addition to the nature of the mathematical content itself, the manner in which this content is approached appears fundamental. In short, we see no interest in "teaching" school mathematics to pre-service teachers. By offering different tasks for the teachers to explore, teacher education practices should not focus on the *teaching* of mathematical concepts but on future teachers' explorations of and engagement with them. We believe it is critically important to engage teachers in a culture and practice of doing mathematics.

Issues of mathematical practices or of a culture of mathematizing as a practice (Bauersfeld, 1998) requires a shift, as Burton (2004) suggests, from mathematical knowledge to mathematical knowing; from mathematics as an object-oriented discipline for someone to know to mathematics as something that one does. Participants in a culture of mathematics are seen as authors and producers of mathematical knowledge, understandings and meanings. In the construction of such a culture, where concepts and ideas are explored and worked on, participants are encouraged to generate ideas, questions and problems, to make explicit and share understandings, to develop explanations and argumentations, to negotiate meanings, to share and explore different ways of understanding problems, concepts, symbolisms, representations as well as solutions and strategies, and to assess and validate other's understandings and explanations (Bartolini Bussi, 1998; Bednarz, 1998; Cobb & Yackel, 1998; Voigt, 1985, 1994). In addition, specific aspects can be highlighted as being fundamental elements of a culture of mathematics:

- The practice of communication. Central to a mathematics-producing practice are participants who are engaged in explaining, discussing, arguing, and validating understandings and meanings (Devlin, 2004; Hersh, 1997; Krummheuer, 1995; Lakatos, 1976);
- The role, relevance and development of language and symbolism in mathematics that are used to express understandings, explanations, arguments, and so on. Symbolism, and its creation, plays a major role in mathematics and in mathematical thinking (Bednarz, Dufour-Janvier, Poirier & Bacon, 1993; Byers & Erlwanger, 1984; Byers & Herscovics, 1977). [In addition, analogies and metaphors can be important in linking some understandings and creating new ones (Bauersfeld, 1998; Lakoff & Núñez, 2000)];
- The role given to errors and how they are handled. Errors play and have played a fundamental role in the emergence of mathematical thinking and understanding; they permit new ways of seeing and understanding and lead to avenues not yet thought of and often unpredicted (Hadamard, 1945; Hersh, 1997; Lakatos, 1976);
- The solving and posing of problems. Doing mathematics is an activity of posing and solving problems of many sorts (Bkouche, Charlot & Rouche, 1992; Devlin, 2004; Hersh, 1997; Lang, 1985). As well, any explorations of mathematical ideas, concepts and situation, or attempts at making sense and developing meanings and explanations for concepts, ideas and procedures are instances of problem posing and solving.

Illustrations of this sort of work and activities can take different forms, most of them coming from work on solving problems. An example of such can be taken from Bednarz (2001) where student teachers are placed into a situation of algebraic generalization and have to (1) extract patterns from a context, (2) formulate this pattern in words for others to understand, (3) symbolise this message using algebra and various accepted and negotiated symbolisms, and (4) validate this message with peers. Another example can be taken from Schmidt & Bednarz (1997) and Proulx (2007a) in which teachers engage with traditional algebraic word problems (of 7^{th} to 9^{th} grade) and have to solve them without using any algebra, being required afterwards to share their solutions as well as compare them with other strategies and solutions obtained by others[3].

It is these sorts of mathematical experiences that we see as being a fundamental part of our teacher education programs in order for student teachers to develop as mathematics educators.

Concluding remarks

We believe these aspects bring a rethinking of the sorts of mathematical experiences and practices that teachers could/should be exposed to through mathematics teacher education practices. We are well aware, as various colleagues have expressed to us, that these orientations appear provocative as well as counterintuitive, since they question well-established accepted structures of teacher education as well as the disciplinary content of academic mathematics as suitable knowledge for *teachers*[4]. These ideas suggest an important shift concerning the mathematical learning opportunities offered to teachers in mathematics teacher education practices.

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[3] Because of space constraints, and because the research is under way, we cannot offer illustrations of how teachers interact with these sorts of tasks (one can however consult the studies referred to). But, we envision for the presentation to offer an example of a task that covers both aspects of culture and of school mathematics and how teachers have engaged with it and interacted, in order to make more concrete aspects of school mathematics and of mathematical culture (e.g., role of communication, of errors, etc.).

[4] Remember, however, that at a personal level this knowledge is fine, but we question its relevance and implications for the constitution of the schoolteachers' professional ways of knowing and practicing mathematics in schools.

This said, we emphasize the fact that this is a theoretical discussion with orientations for which we still need to know a lot more about. Important research needs to be undertaken to better understand what these orientations can provide student teachers with, as well as to know what these approaches could look like. We also need to know more about pre-service teachers themselves in these environments: What sort of knowledge do they develop? What sort of practices do they engage in and develop? In what ways their mathematical knowledge is enhanced and what do they learn? Additionally, we need to know more about how teachers make sense of and interpret these teacher education experiences, as well as how they re-invest these experiences in their teaching practices. Precise attention and analysis of pre-service teachers' interpretation and re-investments of these experiences would in return lead to a better understanding of the teacher education experiences themselves.

Our own research interests and efforts are invested in creating and offering alternative approaches that attempt to offer teachers these experiences of entering a mathematical culture and enriching their knowledge of school mathematics. It is through this shift in thinking and these alternative experiences that we feel teachers will have the opportunity to continue growing in mathematics and enhance their ways of professionally knowing, practicing and teaching mathematics.

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