THE MULTIMEANINGS OF EQUATION:
Considerations concerning its potentialities in teaching and learning of Algebra.

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ABSTRACT

The present work discusses different meanings that can be given to the notion of equation in teaching and learning of Mathematics and their potentialities in the construction of mathematical knowledge. It is a theoretical work, as it summarizes and amplifies the arguments presented by the author's doctorate thesis (Ribeiro, 2007). Grounded in the epistemological study developed in such thesis, it is presented, as can be seen from history of Mathematics facts analysis, different ways of understanding equation. Besides, Research studies on the subject have also been analyzed, showing other ways of explaining the notion of equation. The author validates such results with Semiotic Representation Registers Theory (Duval, 1993, 2003). Afterwards, it is presented and discussed possibilities and potentialities that multimeanings of equation approach can bring to Algebra teaching and learning processes. Among the final conclusions it is pointed out the importance of discussing Elementary Education subjects in teacher training courses without doing it as a simple review of mathematics contents. It is also discussed the relevance that the articulation of multimeanings of equation can provide to teacher and students´ conception enlargement on the notion of equation.

Keywords: Equation. Algebra Education. Multimeanings of Equation. Epistemological Study. Teachers Educational Training.

Introduction

The present paper discusses that different meanings can be given to the notion of equation in teaching and learning of Mathematics, and its potentialities in the construction of mathematical knowledge. It is developed and grounded in the results of the doctorate thesis "*Equation and its multimeanings in mathematics learning: contributions of an epistemological study",* by Alessandro Ribeiro, presented in May 2007, at Pontificia Universidade Católica, São Paulo, Brazil.

This paper was developed under the perspective of a theoretical work, once resumes the discussing approached by Ribeiro (2007) amplifying it to discuss the possibilities and

potentialities of using the different meanings of the notion of equation that can bring to Algebra teaching and learning process.

We perceive through the experience as a teacher and researcher that most of the times we call the idea of equation wherever it is as a tool intra or extra mathematics or as a study object, it gives an excessive emphasis on the procedures and techniques of the solving of the equations damaging other kinds of discussion which may help this knowledge construction in a wider way.

In order to confirm the conclusions about the emphasis that is given to the algebraic transformation when working with equations, I present next some research studies in the mathematics education area which has motivated myself and it has called my attention to the importance of the discussion I am proposing.

Kieran (1992) points out the need about not to allow students in spending much time understanding and conceiving algebraic expressions and equations as being a large amount of letters and symbols where we operate only with numbers – which is called Algebra process aspect – but we have to make them realize that it is possible to understand those entities as objects we can manipulate and in which we can make a lot of other operations beyond to add, to subtract, to divide or to multiply.

Still in order to justify my concerns about the emphasis given to the procedures working with equations, I present the results of my master's degree research study Ribeiro (2001). In that work, I could realize a low performance among 13-14 year-old students even though when they were working on situations which requires a manipulation technique and "mechanical" procedures in equation solving only. The results obtained at that time made me think about inefficiency in the instrumental point of view, still having a super appreciation about the procedure aspect.

In relation to teachers it is observed many times their perceptions of equation also excessively appreciate the procedures. The research study Attorps (2003) developed with 10 high school teachers about their conceptions of equation gives us relevant results to the problem I am constructing. In her work *Teachers' images of the 'Equation' Concept* she presents in its results the insecurity some teachers have stating, for example, $2x + 5y = \sqrt{a}$ it's an equation because they don't know how to find the solution. It also presents most of the teachers have an equation concept very attached to the procedure question. She still says during the interviews teachers frequently related their equation concepts to the way they had

learned – their experiences as students – in order to work with the process of equations solving.

In Lima (2007), the author investigated the meanings given by high school students to the equation and its solving methods. From questionnaires and interviews involving linear and quadratic algebraic equations, the author can notice the investigated students give the meaning of equation as a "sum" to be solved where the equal sign assumes an operational character only. She has also noticed the students can not answer "satisfactorily" the question: " What's the meaning of equation?", presented in one of the instruments.

The considerations made by the author, other previously discussed research studies and others analyzed by myself, such as Cotret (1997), Dreyfus & Hoch (2004), gave us the chance to realize that the notion of equation is understood and confused as being its own solution in many times. Students and teachers give a disproportionate importance if we compare it to other forms of explaining such notion to the technical procedures of the aspects searching for the solution of the equations.

In that way, the presented discussions in this work have the planning to contribute to the enlargement of such a view and those interpretations about the notion of equation once it presents different meanings which can be related to the notion of the equation when Algebra is being learning and teaching.

The appearing of the different meanings of equation

Aiming to drive the discussions to the main purpose of this present article which is ***the presentation of different ways to understand the notion of equation***, I believe it's important to write about the environment which those different meanings appeared during the development of the research in which this paper has been based.

In Ribeiro (2007), the author presents the epistemological development of the notion of equation and after that analyses textbooks having the purpose to identify the ways of how the notion of equation has been understood and constructed during the Mathematics history. Due to the angle of approach and the limitations referred to the present work, the analysis made in the related books will not be here in discussion. However, I'll use such books to exemplify some situations where I recognized signs of multimeanings of equation.

The historical trajectory used by the author begins with the Babylon, Egyptians, and Greeks. From these civilizations, I could get the following conceptions:

> The **notion of equation** used by those civilizations mainly by the Egyptians had **a pragmatic character** and tried to equal quantities with the purpose of finding the unknown quantity value from an **intuitive form** of doing it. (RIBEIRO, 2007, p.55)
> (…)
> On the other hand, the Greeks weren't trying to solve equations which had their origin on practical problems. The **notion of equation** used by Greeks was of a **geometrical character** and in a **deductive form** their solving stood by geometrical manipulations as we realize on the proportion methodology for instance. (RIBEIRO, 2007, p. 60).

Continuing his epistemological study, Ribeiro has investigated how Arabians and Hindus used to understand that notion, observing:

> (…) The issues investigated by Arabians and Hindus seem to give to the notion of equation more and more **algebraic character** .The catalog of expressions which is possible to solve is beyond the specific – built by the accumulated solved problems, particular solving techniques and procedures – to a **catalog** of **all** possible **canonical forms**. (RIBEIRO, 2007, p.67)
> (…) The **notion of equation** used by Arabians and Hindus has presented a more **structural** conception in the meaning of watching the characteristics and properties defined in a class of equations and not in equations related to particular situations. (RIBEIRO, 2007, p.68)

Ending his work, the author shows up how and which were the purposes of European people about using and discussing of the notion of equation. He concludes:

> (…) The **notion of equation** at that period until the solving of cubic and quadric is considered an object of investigation because the operations are made on themselves, **searching** for **general solutions** for those kinds of equations. That is a characteristic which differentiates the way which itself was understood by Babylon or Egyptians for instance. (RIBEIRO, 2007, p.79)

Among the conclusions obtained by the author during the epistemological study, it is worth pointing (…) *from that study at least three different ways of focusing equation emerge: one of them related to a **pragmatic character**, another one related to a **geometric character** and other one related to a **structural character**. (RIBEIRO, 2007, p.82).*

However, besides different ways of understanding the notion of equation which has been observed in this epistemological study, it is realized that in the presented research studies in this work introduction it's identified another way of understanding equation which is the **notion of equation understood as its own solving.** Because, in this way of understanding equation, students and teachers understand the notion of equation as a **process or a technique.**

At last, based on a study made by Chevallard (1991), referring to the educational objects, it has been identified that to the author equation is not a mathematics notion but a paramathematics notion and cannot take place within the educational objects. In that study, it has been observed that one of the arguments which are assumed to the Chevallard posture seems to be the fact it is not possible to define equation excluding it though from the "category" of mathematics notion and, as a result of that, from the educational objects.

However, considering all the meanings identified by Ribeiro (2007), there seem to be of great need approaching the notion of equation in Mathematics teaching and learning processes, in opposite to the ideas presented by Chevallard. As so, in Ribeiro (2007) another way of understanding the notion of equation is presented, besides those which emerge from the epistemological and bibliographic study.

In this other way to conceive the notion of equation, it is dismissing the need of defining itself and therefore be treated as a mathematics notion and that way it can take place with the educational objects, in the educational process. I understand that way I could let myself go from the paradox which had occurred to me during the development of the doctorate thesis: if, on one hand, equation is not a mathematics notion because apparently we cannot define it and cannot be among educational objects; on the other hand all the given meanings to it and its intra and extra mathematics importance point out to the need of considering this notion in Mathematics teaching and learning processes.

Finally, after the construction of this scene, which tried to present the justification and the relevance to the presented paper, and, on other hand, to discuss the sources from the different meanings noticed to the notion of equation in Ribeiro (2007), I introduce the ***multimeanings of equation*** which I propose to be in mathematical teaching and learning procedures.

The multimeanings of equation

The multimeanings presented in this work were conceived and widely discussed in the doctorate work previously appointed. My mainly motivation in the present paper is to emphasize the potentialities the multimeanings of equation have to construct the mathematical knowledge in Algebra teaching and learning process.

The presentation of the multimeanings in this paper will obey an historical order in which in my reading they were appearing and being implicit or not used in the history of Mathematics. I will exemplify each one of them turning to the history of Mathematics, in Mathematics education research studies and Mathematics textbooks.

Before beginning the presentation of the different ways to conceive the notion of the equation – that I choose to call ***multimeanings of equation*** – I think it is relevant to point the differences out sometimes can be very subtle and there is a little difference between them. So let's present those multimeanings:

1. **Intuitive–Pragmatic:** In this meaning the notion of equation is understood as an intuitive notion connected to the idea of equality between two quantities. Its use is related to the resolution of practical problems which are from the day-by-day situations. Here are some examples of situations which characterize that meaning:

    - Babylon and Egyptians: practical problems involving agriculture issues for instance;
    - In the textbook by Bourdon (1897).

2. **Deductive-Geometrical:** In this meaning the notion of equation is understood as a notion connected to geometrical figures and segments. Here are some examples which characterize that meaning:

    - Greeks: using the proportion methodology and the area application. The proportion methodology allows to construct a line segment x given by $a : b = c : x$ or $a : x = x : b$, where a, b, c are line segments. Related to the method of areas application we can turn to the *Euclidean elements*, using the proposition 44 from the book I, and the propositions 28 e 29 from the book VI;

- Curves geometry: Khayyam found a geometric solution to the cubic equation $x^3 + ax = b$, using the circle line intersection $x^2 + y^2 = qx$ and the parabola $x^2 = py$ and also worked with a cubic $x^3 = ax + b$, using parabola intersection $x^2 = \sqrt{a}y$ and the equilateral hyperbola $x\left(\frac{b}{a} + x\right) = y^2$

3. **Structural-General**: In this meaning the notion of equation is understood as a defined structural notion and with its own properties and characteristics. The equation presented in this case is considered as itself working on itself on the search for general solutions to an equation class from the same nature. Here are some examples of situations which characterize that meaning:

    - al-Khwarizmi: although the equations which he used to work were from practical problems, his attention was focused to the determination of the solving of any quadratic equation. He established two important operations *al-jabr* and *al muqabalah* which reduced the equations treated by him in six types in its canonical form.
    - Descartes: when the using of his Cartesian Method begins to take the own equations not as a way of organizing phenomena, but as a field of objects which need new forms to its organization: it would be the solution of the equations using the canonical form;
    - Other Mathematicians from Descartes such as Abel and Galois who started to investigate the structure of the equations solving process, aiming to find or to show that there wasn't an algorithm which was able to solve the equations with degrees superior to fourth order by radical identities.

4. **Structural-Group**: In this point of view, the notion of equation is understood as a structural perspective which is directly connected to the notion of group. It is seen as a tool in order to solve problems which are related to groups. Some examples of situations which characterize that meaning can be found in Bourbaki (1970), Rogalski (2001) and Warusfel (1969).

5. **Processing-Technicist**: It understands equation as its own solution – such as the methods and techniques used in order to solve them. Differently from structuralists, they don't see the equation as a mathematical entity in which the operations and manipulations which are carried out answer to defined rules. Some examples of situations which characterize that meaning can be found in research studies carried out in the Mathematics Education area such as in Cotret (1997) or in Dreyfus & Hoch (2004)

6. **Axiomatic-Postulating**: It understands equation as a notion of Mathematic which doesn't need to be defined. An idea from other ideas are constructed, mathematical and non mathematical. By this concept, the notion of equation is used in the same meaning of Primary notion, such as point, line and plane in the Euclidean Geometry. One example about this meaning can be observed from the works by:

    - Chevallard, who from an indirect way conceives equation with that meaning in his work about Educational Transposition when he refers to the mathematical and paramathematical notions because the notion of equation cannot be understood as a mathematical notion because it doesn't have only one definition, by the way it doesn't have to, after all it is a paramathematic notion used as an auxiliary knowledge when we work with some mathematic notion itself.

This last meaning presented, the axiomatic- postulating, in my point of view must be considered as the first of them to be discuss in Algebra learning and teaching process. We don't need to worry about defining the notion of equation with it; we can give some priority to the discussion of the notion of equation main idea, whatever it is the equality idea, allowing us to integrate that meaning to others which has been introduced before.

Conclusions and final considerations

I bring in this part of my work some conclusions and orientations that I believe are relevant and important to Algebra teaching and learning, mainly in the teachers´ training courses, where I understand the approach of the multimeanings of equation gets relevance.

Because of the diversified way of understanding the notion of equation, I think about discussing the needs of working from different multimeanings in a full way trying to relate one meaning to another. Such proposal is confirmed by the discussions aimed by Duval (1993, 2003) in his Semiotic Representation Records theory once this author defends the articulation importance of the different semiotic representation records in the understanding of Mathematics.

Accepting the importance of using different semiotic representation records to the construction of the mathematical knowledge, I aim the hypothesis that articulating the intuitive-pragmatic with the deductive-geometrical, for instance, we can give situations which the idea of equation is still understood as a problem between quantity equalities and it can be interpreted and represented from different graphic forms, whatever it is by diagrams, graphic planes or even though by the intersection of two curves generating the solution to the presented problem.

Another hypothesis I point out, among many possibilities which can be exploiting from the using perspective of the multimeanings of equation, *are the articulations between the deductive-geometrical and the structural-group. Working with situations involving equations where the unknown quantity is not a number but functions we can discuss about other kinds of equations such as the differential equation, having in view the main idea of equation which it is the equality.* In that meaning we are able to exploit the idea of equality, extending the students´ conception have from equation to beyond of *what you must find is the unknown number*.

Considering the value ideas in Duval's theory where *we know a mathematical object when we are capable of interpreting it and understanding it by different semiotic representation records*, I believe the classroom work, in an articulate way, where the multimeanings of equation discussion is considered, will serve as starting point to a larger significant study of such important algebraic knowledge. In my point of view, this perspective has the possibility to enlarge the conception of equation beyond the "simple" rules and manipulation techniques.

Therefore, nowadays I continue my work developing new researches in order to elaborate, apply and evaluate the impacts than learning situations considering the multimeanings of equation can have among the students and teachers of Mathematics. Those situations give the privilege to articulate those different meanings, considering the learning level and the offered purposes to Mathematics Education I would like to practice.

Research studies as the one developed by Ribeiro (2007) and discussed in the present paper can bring to the environment of the Mathematics teachers training courses a discussion about Elementary Education subjects - which is the equation's case - that does not considers a simplify review character and resume the "primary" contents, but make possible epistemological and didactical-pedagogical discussions about those elementary knowledge helping the conceptions enlargement the future teachers and the in-service teachers may have about those mathematical notion.

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