MATHEMATICAL SYMBOLIZATION:

CHALLENGES ACROSS LEVELS

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Symbolism is a major dimension of the language of mathematics at all learning levels. It is a tool for expressing relationships and for problem solving. Accessing and becoming fluent with symbolization is vital for mathematics success. But symbols entail many challenges for students. These challenges run the gamut from reading aloud (verbalizing) through understanding and writing. Teachers at all levels need to be sensitive to students' challenges and have strategies for supporting students in gaining fluency. This paper identifies several general categories of challenges in each of the three areas mentioned, a few examples in each category, and references for supporting teaching and learning the symbolic language of mathematics.

INTRODUCTION

Symbolism is a major dimension of the language of mathematics at all learning levels. Young children learn symbols for numerals and how they work in the base ten place-valued system. Older children learn notations for common fractions and decimal fractions. Later, students learn variables in algebra and an array of notations in geometry, probability, statistics, and combinatorics. The list continues with calculus and more advanced topics. We are fortunate that our discipline commands such a concise, flexible, and powerful notational system for communication and problem solving. Pimm (1991) notes that symbols illustrate the structure of mathematics, help make manipulations routine, enable reflection about mathematics, and facilitate compactness and permanence of thought.

But symbols entail many challenges for students. These challenges run the gamut from reading aloud (verbalizing) through understanding and writing. While there is software that will perform symbol manipulation for users, students' having meanings for symbols and using them productively is still a linchpin of mathematical success. Teachers at all levels need to be conscious of the challenges that symbolization presents to students and have strategies for supporting students in gaining fluency. This paper identifies general categories of challenges in each of the areas of verbalization, understanding, and writing with a few examples in each category. Several sources for teaching strategies are provided at the end.

VERBALIZING SYMBOLS

Usiskin (1996) noted, "If a student does not know how to read mathematics out loud, it is difficult to register the mathematics." How many of us recall being stuck when some foreign symbol appeared as we were reading a mathematics text? Simply being able to *verbalize* symbols, while a surface feature of language (Pimm 1987), is a step in learning mathematics as well as a communication tool. But there are many challenges for students.

- *Verbalization of some symbols uses both conventions and meanings*. For example in the base ten place-valued system, 1700 is both 17 hundred and 1 thousand 7 hundred (as well as 170 tens and others). With decimal fractions, students are often told, for example, to read 0.72 by saying what appears to be a whole number and appending the name of the rightmost decimal place. This method may be learned as a "rule" in which case renaming the symbols to 0.720 (seven hundred twenty thousandths) is an application of the same rule and may not be identified as a mathematical equivalence.
- *Verbalization of symbols may be superficial or meaningful*. For example, adults may read 2.08 as "Two point oh eight" but we want students to know and read with meaning, "two and eight hundredths." Students who persist in using "point" rather than place value language, struggle with interpretations and comparisons, and later with operations with decimal fractions (Sowder 1997). In work with fractions, English speakers will often read ¾ as "three over four," referring literally to the symbols rather than to the meaning of "three fourths." Similarly, some people say "three out of four" suggesting that the measures are discrete when they may very well be continuous (Siebert & Gaskin 2006). Both of these uses ignore the meaning in which the denominator (fourths) names the size of the unit of some whole and the numerator measures the number of those units. In algebra, when the opposite of x, $-x$, is read as "negative x" students are misled into thinking the overall value of the expression must be negative. Similarly when r^2 or r_2 is read superficially as "r two," the mathematical meaning or intent is masked. The same problem arises when we talk students through the use of calculators. For example, for the expression $3(x + 2)$ some people say, "Three left paren x plus 2 right paren" when the meaning is "3 times the sum of x and 2." In calculus we often read dy/dx literally as "dee y dee ex" when students need to understand that the symbols represent "the derivative of y with respect to x."
- *Some symbolic expressions may be verbalized in multiple ways*. Fluency requires being able to translate among variations. Furthermore, these variations change over time. For example, in arithmetic 10 – 7 is "ten minus seven," or "ten take away seven," but in algebra and beyond we want students to recognize subtraction using other phrases, so $x - y$ may be read as "the difference between x and y" or "the distance between x and y." These differences of phrasing may be transparent to teachers, but need attention for students.
- *Unlike many alphabetic languages, symbols are not always read in one direction*. So, for example, expressions involving summations or rational

expressions need attention. In place value, in contrast, while numbers are read from left to right, determining how to begin to read the left end of a large whole number requires attending to the decimal point and figuring out places from there.

UNDERSTANDING SYMBOLS

Issues of verbalization refer to the surface of symbols. What we really want is for symbols to hold meanings for students. Why is meaning making or *understanding* challenging? Here are some perspectives.

- *The same symbol may have different meanings*. For example, a small dash may mean "opposite," "minus," or "negative." Parentheses as in (-2,3) can represent an ordered pair or an open interval on the number line. The raised -1 symbol can mean an inverse function or a reciprocal. The prime symbol may mean the complement of a set, feet, minutes, or a derivative. In more advanced mathematics, a variable may mean a value, a set, a point, a vector, or some other abstraction. In general, we expect students to use context (as they do in reading verbal text) to support their making sense of the meaning of a symbol in a particular setting.
- *Multiple symbols may represent the same concept*. Division, for example, is represented in several forms. Division in arithmetic is often symbolized with $12 \div 3$ or with box notation. As students transition into algebra, a major shift is the more common representation of division as a fraction, $\frac{12}{3}$. Other cases of multiple symbolizations arise in multiplication e.g., 3 x 4, (3)(4), 3*4, or 3•4; differentiation, e.g., dy/dx, y', or $f'(x)$; combinatorics e.g., $C(n,r)$, ${}_nC_r$, or $\binom{n}{r}$; and other areas.
- *Symbols may be implicit, but central to understanding*. A major hurdle for young children is our place-valued notation system that involves invisible products and sums: 234.56 = 2 x 100 + 3 x 10 + 4 x 1 + 5 x 1/10 + 6 x 1/100. In initial work with decimal fractions, it is not obvious to students that 4 = 4. = 4.0 = 4.00 and so forth. In work with mixed fractions students need to recognize implied additions, like $3\frac{1}{2} = 3 + \frac{1}{2}$. In algebra we use implied multiplication, as in $3x$. There are even cases where there is no operation, but mathematically one might be imposed. This insight is a major problem solving tool, for example, $x = 1x$, $x = x + 0$, or $x = x^1$.
- *The placement or ordering of symbols may affect their meaning*. Again, our place value system is predicated on invisible products (digit times its place's value) and sums, and the multipliers are dependent on the location of a digit in relation to the decimal point. The decimal point (or comma) is subtle and imposes considerable information (orders of magnitude!), but coming to

appreciate this is not trivial. In algebra, $xy = yx$, but xy and yx may represent different variable names in a computer algebra system. A point may be part of the decimal system or, when raised, may indicate multiplication, as 5.6 versus 5·6. Some expressions require applying the convention of order of operations. For example, -3^4 means the opposite of the fourth power of 3 (equals -81) while $(-3)^4$ means the fourth power of negative 3 (equals 81). Three major sequences have symbolizations that differ only in the placement of the symbols: $t_n = 2n$, $t_n = 2^n$, and $t_n = n^2$. While mathematics naturally attends to commutativity with respect to operations, we need to attend to this ordering issue of symbols, as well, when supporting students' symbolization learning.

- *Operations may be implicit but students feel they must be executed*. A symbol like $\pi/4$ or $\sqrt{2}$ may be a perfectly meaningful expression to a mathematician, but students ask, "What is it really?" believing that until a calculation is completed and some decimal value produced that the expression is unfinished or unacceptable. Coming to see implicit, unexecuted operations as sufficient is a stage in mathematical symbolism maturity.
- *Specific variables may be associated with different contexts*. The same formula with different letters is used in different contexts:
  - $a^2 + b^2 = c^2$ Pythagorean Theorem.
  - $x^2 + y^2 = r^2$ Formula for a circle centered at the origin.
  - $x^2 + y^2 = z^2$ Square of absolute value of a complex number.

  These are nuances of usage to which students need acculturation.
- In introductory algebra, x, y, and z are typically used as unknowns; m is used for the slope of a line and b for the y-intercept. In contrast, in statistics, a is often the y-intercept of a regression line and b is the slope. In algebra and calculus, f and g are common names of functions.

PRODUCING SYMBOLS

As with language literacy, reading and writing are related processes, but each requires specific attention. When students need to write mathematical symbolism, either for communication or problem solving, new issues arise.

- *Students sometimes use incorrect syntax when translating into technology*. For example, implicit groupings may need to be made explicit: $\sqrt{9 + 16}$ must be entered as √(9 + 16) or $\frac{x^2 - y^2}{x - y}$ must be entered as $(x^2 - y^2) \div (x-y)$. Also, in using technology, the order of symbols may be different than on paper, e.g., the division $3\overline{)12}$, is entered as $12 \div 3$.
- *Students over-symbolize*. U.S. groceries are notorious for mis-writing money, as when twenty-five cents is written 0.25¢. This may correlate with

the conjecture that people do not see cents as a fraction of a dollar, but as another system of whole numbers (Zawojewksi 1983). For exponentiation students have been known to include both the technology and paper formats and write 5^^{3}. Some algebra students write equal signs between equations, producing false run-on sentences. So, for example, in evaluating $11(5) + 3$ students write $11(5) = 55 + 3 = 58$, recording (inappropriately) what they might be thinking mentally.

- *Students misunderstand when distribution is or is not appropriate*. In general for a function f, $f(a + b) \neq f(a) + f(b)$, even though for a variable k, $k(x + y) = kx + ky$.

TEACHING STRATEGIES

In thinking about strategies to help students avoid these difficulties it is useful to distinguish symbolization issues that are syntactic and those that are semantic. When what is to be learned is syntactic or a matter of convention, then we need to be careful to show the new symbol, demonstrate how it is written and used, and give students a chance to say it, read it, write it, and practice its use. When (more often) the issue is semantic, i.e., related to mathematical meanings, then the symbols need to be attached to referents that are already meaningful to students (Wearne & Hiebert 1988). Students need to be engaged in a variety of meaning-making activities: appreciating the purpose of the symbolization, thinking aloud so teachers may understand how students interpret symbols; translating symbols into words, diagrams, word problems, or other better understood symbols and the reverse of these processes (Goldin 2003); confronting theirs and other students' errors and "debugging" those productions; contrasting similar but distinct expressions; and others. Many sources provide teaching suggestions in this vein (Greenes & Rubenstein 2008; Thompson et al. 2008; Kenney et al. 2005; Rubenstein and Thompson 2001; Reehm 1996; Siegel et al. 1996; Shuard & Rothery 1984; Kane Byrne, & Hater 1974). The first step is teachers who recognize that the way mathematics is encoded, as clear as it may be to the mathematically literate, is a new language to students, holds many challenges for them, and is worthy of more specific instructional attention.

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