ICME 2008

Multiple representation tasks that foster conceptual development

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Design-based research

- Design-based research is about transforming educational practices in authentic situations
- Analytic sciences - explaining phenomena. Design sciences - determining how designs behave under different conditions.

(Collins, 1992)

Design research

- A product or theoretical artifact is created (e.g. a teaching unit, a model for professional development).
- Multiple theories are called upon in the design and revision of the product.
- The product and the theories are used, reflected upon and developed together, iteratively.
- The researcher acts as an interventionist rather than as a participant observer, in a collaborative, reflective relationship with the teacher.

(Adapted from Woods and Berry (2003))

Design starts with questions...

- What is the context for the design?
- What purpose is the design intended to serve?
- What learning theories are appropriate?
- What empirically tested design principles emerge from these theories?

Context

- In England, the GCSE is the major end qualification for the compulsory phase of education.
- Each year, 16-19 year-old students that fail to attain the minimum grade required for planned careers or entry into higher education embark on re-sit courses within further education (FE) institutions.
- National inspection reports reveal that the teaching on these courses is teacher-centred and transmission-oriented.

Context

- 55% of students fail to attain GCSE grade C.
- Colleges over-recruit.
- High drop out rates.
- Average level of attendance is only 63%.
- Teaching is ‘narrow and unimaginative’.
- Pass rates poor.

Purpose: to move from transmission....

- **Mathematics is seen as**
  - a body of knowledge and procedures to be ‘covered’
- **Learning is seen as:**
  - an individual activity based on listening and imitating
- **Teaching is seen as**
  - structuring a linear curriculum for the students
  - giving explanations and checking these have been understood through practice questions
  - ‘correcting’ misunderstandings when students fail to ‘grasp’ what is taught

.... to collaboration

- **Mathematics is seen as**
  - a network of ideas which teacher and students construct together
- **Learning is seen as**
  - a social activity in which students are challenged and arrive at understanding through discussion
- **Teaching is seen as**
  - non-linear dialogue in which meanings and connections are explored
  - recognising misunderstandings, making them explicit and learning from them

Principles for effective teaching

- Build on the knowledge learners already have
- Expose and discuss misconceptions and surprising phenomena
- Use higher-order questions
- Make appropriate use of cooperative small group work
- Emphasise reasoning not ‘answer getting’
- Use rich, collaborative tasks
- Create connections between topics
- Use technology and other resources in creative, appropriate ways

Purposes of Mathematics Teaching

- **Fluency**
  in recalling facts, performing skills
- **Meaningful interpretations**
  for concepts and representations
- **Strategies**
  for investigation and problem solving
- **Awareness**
  of nature & values of the educational system
- **Appreciation**
  of the power of mathematics in society

Principles from theory...

In coming to understand a concept,
a student must:

- single it out and bring it to the forefront of attention (identify);
- notice similarities and differences between this concept and other similar ones (discriminate);
- identify general properties of the concept in particular cases of it (generalise)
- and begins to perceive a unifying principle (synthesise)

(Sierpinska, 1994).

Principles from theory...

- Students learn concepts by appropriating, internalising and reorganising language and symbols

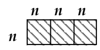
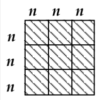
Principles from theory

- Focus directly on significant conceptual obstacles
- Activate and build on prior knowledge.
- Stimulate 'cognitive conflict' to promote re-interpretation, reformulation and accommodation.
- Use rich tasks that are accessible, extendable, encourage decision-making, creativity and higher order questioning
- Use multiple representations to create bridges between concepts
- Encourage students to change roles and explain and teach one another
- 'Production of answers' must give way to reflective periods for examining alternative meanings and methods.

'Genres' of task for developing concepts

1. Interpreting multiple representations
2. Classifying mathematical objects
3. Evaluating mathematical statements
4. Creating and solving problems
5. Generalising existing situations

Interpreting Multiple Representations

	
Square *n* then multiply your answer by 3	Multiply *n* by 3 then square your answer
$9n^2$	$(3n)^2$
$3n^2$	Square *n* then multiply your answer by 9

Mathematical goals of the design

To help learners to:

- **translate between words, algebraic expressions, tables and area representations.**
- **recognise the order of operations**
- **recognise equivalent algebra expressions.**
- **understand the distributive laws of multiplication and division over addition.**

Represent the order of operations ...

- Add six then multiply by two
- Multiply by two then add six
- Add six then square
- Square then add six
- Square then multiply by three
- Multiply by three then square
- Divide by two then add six
- Add six then divide by two

Discriminate between commonly confused expressions

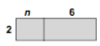
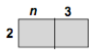
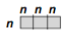
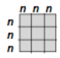
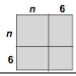
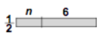
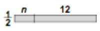
$$2(n+3) = 2n+6 \qquad 2(n+3) \neq 2n+3$$

$$(3n)^2 = 9n^2 \qquad (3n)^2 \neq 3n^2$$

$$(n+6)^2 = n^2+12n+36 \qquad (n+6)^2 \neq n^2+6^2$$

$$\frac{n+6}{2} = \frac{n}{2}+3 \qquad \frac{n+6}{2} \neq \frac{n}{2}+6$$

Interpreting Multiple Representations

$\frac{n+6}{2}$	$3n^2$	Square *n*, then multiply by three	*n*: 1, 2, 3, 4 *Ans*: 14, 16, 18, 20	
$2n+12$	$2n+6$	Add six to *n*, then multiply by two.	*n*: 1, 2, 3, 4 *Ans*: , , 81, 144	
$2(n+3)$	$\frac{n}{2}+6$	Add six to *n*, then divide by two	*n*: 1, 2, 3, 4 *Ans*: , 10, 15, 22	
$(3n)^2$	$(n+6)^2$	Divide *n* by two, then add three	*n*: 1, 2, 3, 4 *Ans*: 3, , 27, 48	
$n^2+12n+36$	$\frac{n}{2}+3$	Add six to *n*, then square the answer	*n*: 1, 2, 3, 4 *Ans*: , , 81, 100	
n^2+6	Add three to *n* then multiply by two.	Square *n*, then multiply by nine	*n*: 1, 2, 3, 4 *Ans*: , 4, , 5	
n^2+6^2	Multiply *n* by two then add twelve	Multiply *n* by two, then add six.	*n*: 1, 2, 3, 4 *Ans*: 6.5, 7, 7.5, 8	

Translations between ...

To From	Words	Algebra Expressions	Tables	Area diagrams
Words				
Algebra Expressions				
Tables				
Area diagrams				

Ground rules for students

- **Give everyone a chance to speak.**
- **Listen without interrupting.**
- **Check that everyone else listens.**
- **Try to understand what is said.**
- **Build on what others have said.**
- **Challenge what is said.**
- **Demand good explanations.**
- **Treat opinions with respect.**
- **Share responsibility.**
- **Try to reach agreement.**

The role of the teacher

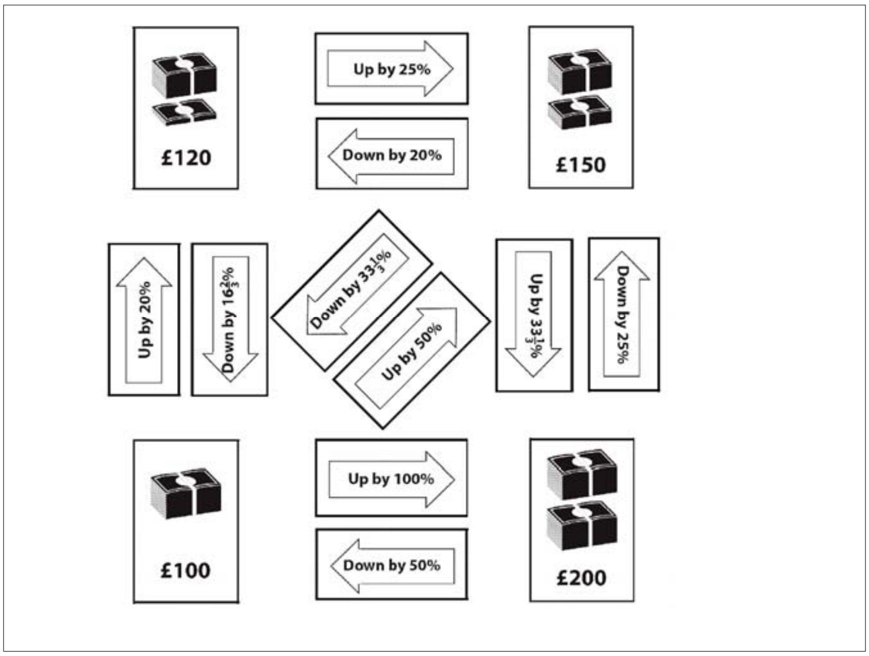
- **Make the purpose of the discussion clear.**
- **Keep reinforcing the 'ground rules'.**
- **Listen before intervening.**
- **Join in, don't interrupt.**
- **Don't judge or praise - this discourages contributions.**
- **Ask students to describe, explain and interpret.**
- **Do not do the thinking for the students.**
- **Don't feel you need to resolve everything before leaving a group or before the end of the lesson.**

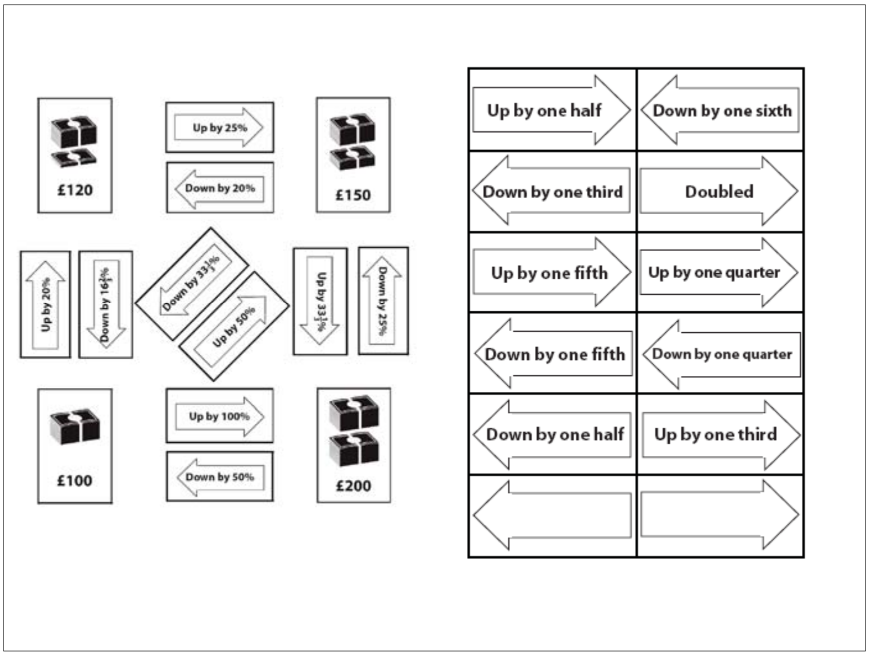
Percentage increase and decrease

In a sale, the prices in a shop were all increased by 25%.

There was then a sale in which they were all reduced by 20%.

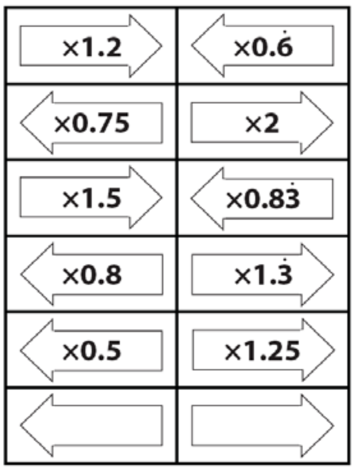
What was the overall effect on the shop prices?
Explain how you know.


£120
Up by 25%
Down by 20%
£150
Up by 20%
Down by 16⅔%
Down by 33⅓%
Up by 50%
Up by 33⅓%
Down by 25%
£100
Up by 100%
Down by 50%
£200


£120
Up by 25%
Down by 20%
£150
Up by 20%
Down by 16⅔%
Down by 33⅓%
Up by 50%
Up by 33⅓%
Down by 25%
£100
Up by 100%
Down by 50%
£200
Up by one half
Down by one sixth
Down by one third
Doubled
Up by one fifth
Up by one quarter
Down by one fifth
Down by one quarter
Down by one half
Up by one third

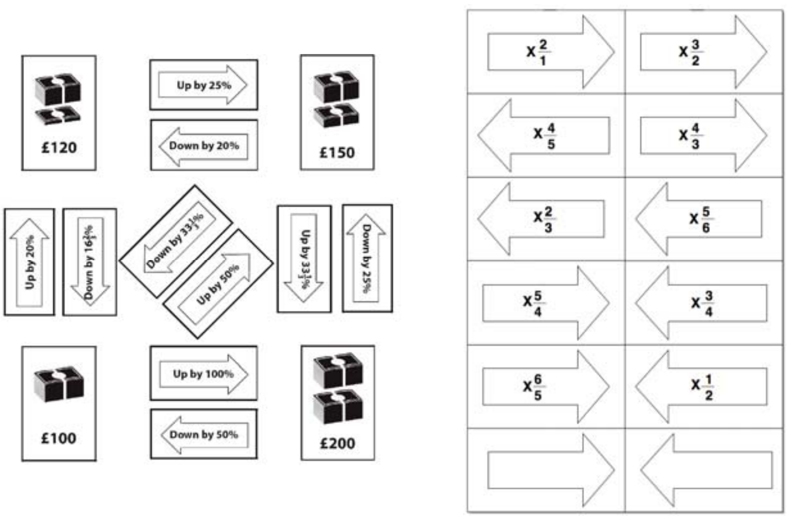
Connections made with inverses ...

Doubled	**Down by one half**
Up by one half	**Down by one third**
Up by one third	**Down by one quarter**
Up by one quarter	**Down by one fifth**
Up by one fifth	**Down by one sixth**
....	
....	



Connections made with inverses ...

x 2	x 0.5
x 1.5	x 0.66...
x 1.33...	x 0.75
x 1.25	x 0.8
x 1.2	x 0.83



Connections made with inverses ...

$$\times\frac{2}{1} \qquad \times\frac{1}{2}$$

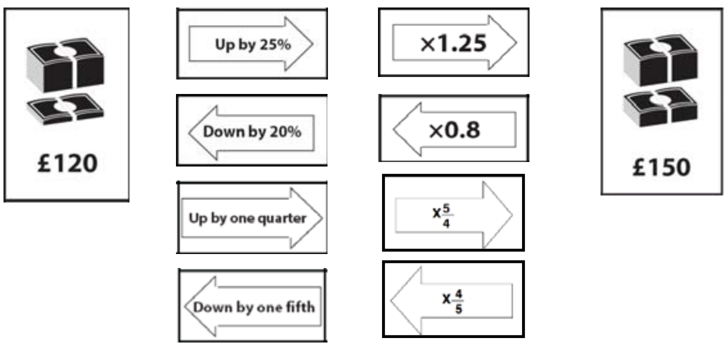
$$\times\frac{3}{2} \qquad \times\frac{2}{3}$$

$$\times\frac{4}{3} \qquad \times\frac{3}{4}$$

$$\times\frac{5}{4} \qquad \times\frac{4}{5}$$

....

Alternative representations

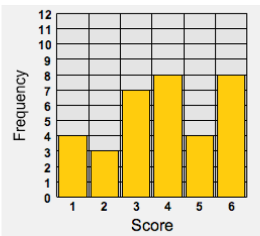


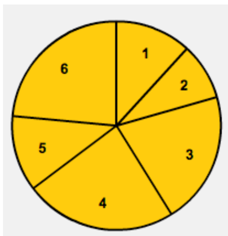
Translations between ...

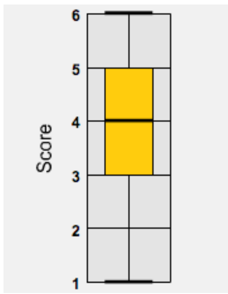
To From	Words	Percents	Decimals	Fractions
Words				
Percents				
Decimals				
Fractions				

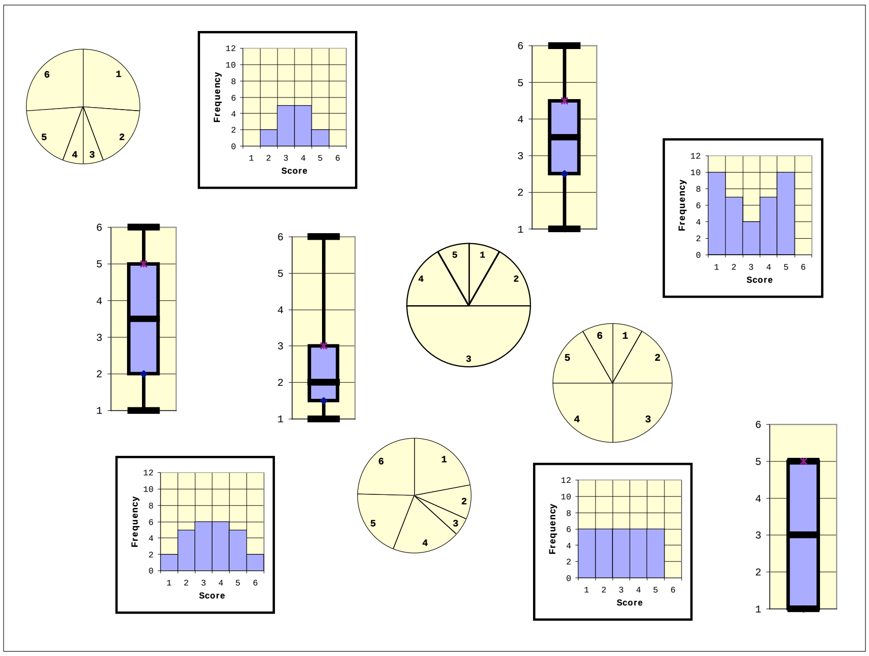
Penalty kicks

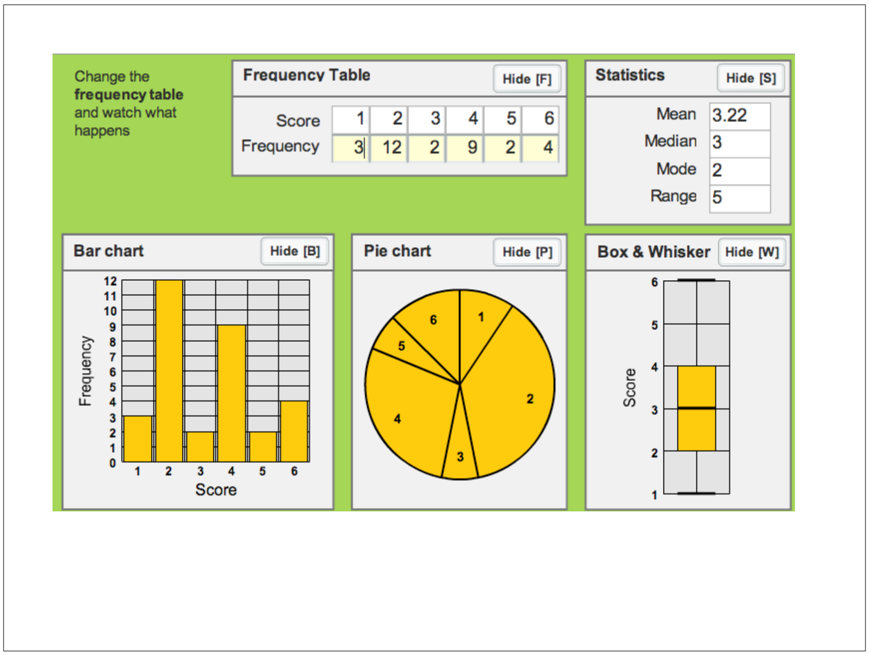
These charts represents the scores that were obtained when a number of people entered a penalty-taking competition. Each person was allowed six penalty kicks.

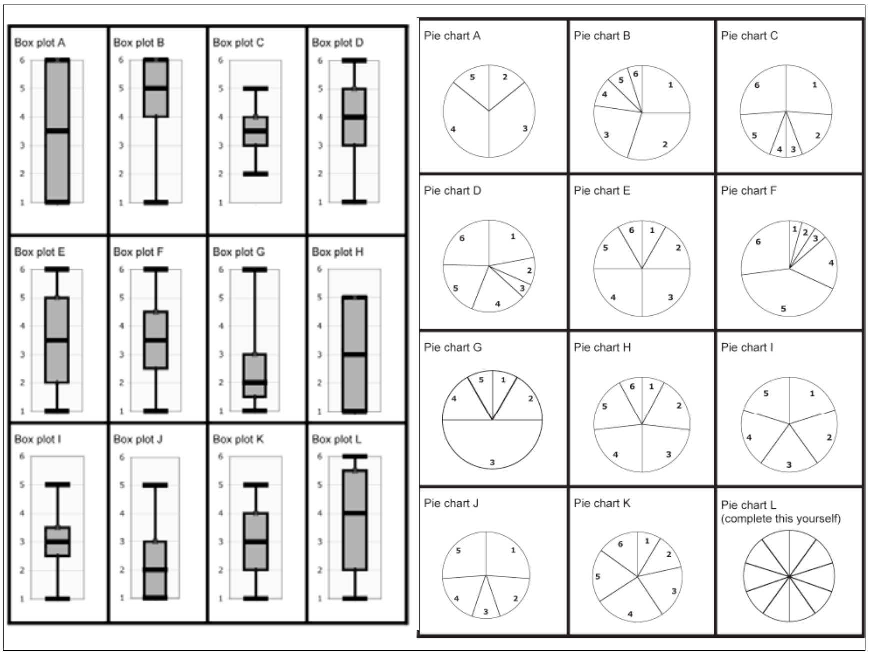







Frequency
Score


Change the **frequency table** and watch what happens
Frequency Table
Hide [F]
Score 1 2 3 4 5 6
Frequency 3 12 2 9 2 4
Statistics
Hide [S]
Mean 3.22
Median 3
Mode 2
Range 5
Bar chart
Hide [B]
Frequency
Score
Pie chart
Hide [P]
Box & Whisker
Hide [W]
Score



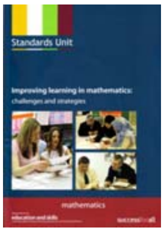
"The good thing about this was, instead of like working out of your textbook, you had to use your brain before you could go anywhere else with it. You had to actually sit down and think about it. And when you did think about it you had someone else to help you along with you if you couldn't figure it out for yourself, so if they understood it and you didn't they would help you out with it. If you were doing it out of a textbook you wouldn't get that help. After I did it I found that I used a lot of brain power, but I felt dead clever. Do you know that when you have actually done something and you actually put all your effort into something.. it makes you feel dead clever. I've told all my friends that I have actually done a bit of work in maths. 'Cause I never thought I was any good at maths, but I was alright with that."

(Lauren, a 16-year-old low-attaining student).

Sources of such tasks

Swan, M. (2005). ***Improving Learning in Mathematics: Challenges and Strategies.***
Sheffield: Teaching and Learning Division, Department for Education and Skills Standards Unit.

http://teachingandlearning.qia.org.uk



Swan, M. (2006). ***Collaborative Learning in Mathematics: A Challenge to our Beliefs and Practices.***

London: National Institute for Advanced and Continuing Education (NIACE); National Research and Development Centre for Adult Literacy and Numeracy (NRDC).