Applying APOS theory as a theoretical framework for collaborative learning in teacher education

Deonarain Brijlall [1] and *Aneshkumar Maharaj* [2]

[1] brijlalld@ukzn.ac.za School of Science, Maths and Technology Education, University of KwaZulu-Natal
[2] maharaja32@ukzn.ac.za School of Mathematical Sciences, University of KwaZulu-Natal

Abstract

This article reports on the use of APOS theory as a theoretical framework in a study which investigated fourth-year students' understanding of the two fundamental concepts monotonicity and boundedness of sequences. Research was done at the Edgewood Campus of the University of KwaZulu-Natal in South Africa. These concepts were taught to undergraduate teacher trainees wishing to specialise in the teaching of mathematics in the FET school curriculum. Worksheets based on an examples and non-examples approach were designed to foster collaborative learning. A group of twenty three students participated in the project. This paper, specifically, reports on the investigation of students' responses based on a learning theory within the context of advanced mathematical thinking and makes a contribution to an understanding of how these students constructed the two concepts in a collaborative way.

Introduction

Many studies, for example (Dubinsky et al, 2005), analysed student mathematical learning on an individual basis. This study however analysed teacher-trainees' understanding, after they carried out investigations first individually and then in a collaborative manner. This is to address the learner-centred approach which underpins Curriculum 2005 (DoE, 2003). We report on an investigation based on the use of worksheets and group-work to construct concepts. To collaborate is to work with another or others. In practice, collaborative learning has come to mean students working in pairs or small groups to achieve shared learning goals (Barkley et al, 2005).

Vidakovic (1996, 1997) used APOS theory in the context of collaborative learning. Those investigations focused on the differences between group and individual mental constructions of the inverse function concept. Vidakovic described the construction processes for developing schema (genetic decomposition) of the inverse function. The main study consisted of computer activities which stimulated the students' steps of constructions in the development of schema. Our work, however, implemented designed worksheets to stimulate the cognitive constructions in the development of schema on monotonicity and boundedness of infinite real sequences.

Wu (2003) argued that pre-service development of teachers for grades 6 to 12 require courses which consolidate, *mathematically,* those topics which do not stray far from the high school mathematics curriculum. In particular, "….*they should revisit all the standard topics in high school from an advanced standpoint, and enliven them with motivation, historical background, inter-connections and above all, proofs*." (Wu, 2003). In work preceding this study it was shown that many of the concepts dealt with in the Real Analysis module; taught at the Edgewood Campus; strengthen the ideas on which the high school mathematics is based (Brijlall, 2005). Such concepts are indicated in the next section. It seems that many students perform poorly in mathematics because they: (a) are unable to adequately handle information given in symbolic form which represent objects [abstract entities], for example mathematical expressions, equations and functions, and (b) lack adequate schema or frameworks, which help to organize and link different objects (Maharaj, 2005).

The above context led us to formulate the following research question: *How does the implementation of a structured worksheet design using APOS theory, to promote collaborative learning, influence the construction of concepts in real analysis?* In particular we look at the construction of the concepts monotonicity and boundedness of infinite real sequences. The structured design used an examples and non-examples approach discussed by Cangelosi (1996). In answering this question we focus on (a) sorting, (b) reflecting and explaining, (c) generalizing, (d) verifying and refining, and (e) extension of generalization. These stages were chosen since we believe they could be exploited to facilitate the following which form the framework for APOS theory and contribute to conceptual understanding: action, process, object, schema.

Background

Knowledge Base for Educators

One of the expectations of the Norms and Standards for Educators (DOE, 1999) is that the educator be well grounded in the knowledge relevant to the occupational practice. She/he has to have a well-developed understanding of the knowledge *appropriate* to the specialism. Many mathematics educators find themselves in a position requiring them to implement the syllabus, which includes certain topics they are unfamiliar with. According to Adler (2002), educators with a very limited knowledge of mathematics need to develop a base of mathematical knowledge. They need to relearn mathematics so as to develop conceptual understanding. Taking this into account we attempted to make certain that trainee-teachers leave with a base of knowledge relevant to their occupational needs. Mwakapenda (2004) concurs when stating that a significant concern in school mathematics is learning with understanding of mathematical concepts.

We show via a few examples the direct relationship of the concepts with topics from the school FET syllabus as suggested by the National Curriculum Statement (DOE, 2003). The following are examples of the types of problems that are covered:

Open and closed intervals

The notion of open and closed intervals play an important role throughout the FET curriculum. For example the solution to the inequality $x^2 - x < 6$ uses an open interval (as $x \in (-2\ ;\ 3)$). The question, solve for θ in $\sin\theta = \frac{1}{2}$ for $\theta \in [0^0; 360^0]$, uses a closed interval. Graduate students confront many other types of open and closed sets. The module questions the openness/closedness of $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$. This should yield a better understanding of the number system as well.

Sequences and Series

The learning outcome in the National Curriculum Statement (2003) prescribes the investigation of number patterns culminating in arithmetic and geometric sequences and series. Learners in grade 12; from 2008 onwards; are expected to interpret recursive formulae (eg. $T_{n+1} = T_n + T_{n-1}$). When proving $\sqrt{2}$ irrational an approach using recursive formulae is employed. This prepares the teacher trainee, as recursive formulae were not covered in the old syllabus. Our undergraduate students therefore need to be well versed in sequences and series. The realization that a real sequence is merely a function whose domain is the set of naturals is fundamental. Deeper understanding into the concepts of convergence and divergence of both sequences and series are dealt with in the Real Analysis course. This will aid student teachers when dealing with the convergence of geometric series in grade 11. Students are also allowed the opportunity to investigate ideas and tasks to give a wider perspective to arithmetic and geometric sequences.

Monotonicity

A sub-skill in Learning Outcome 2 (DOE, 2003) for grade 12 is to identify the intervals on which the function $y = a^x$, $a > 0$ and its inverse $y = \log_a x$ increase or decrease. These ideas are consolidated when studying monotonicity. Formally, a function *f* is called a monotonically increasing function on $[a;b]$ if $f(x) > f(y)$ whenever $x > y$, $\forall x, y \in [a;b]$ and likewise one can define a monotonically decreasing function on an interval. A function is called monotonic if it is either monotonically increasing or decreasing.

Theoretical Basis

Piaget, cited in Bowie (2000), expanded and deliberated on the notion of ***reflective abstraction***. He regarded this to be a major instigating factor for the development of mathematical cognition, and distinguished three types of abstractions: (a) Empirical abstraction, (b) Pseudo-emperical abstraction, and (c) Reflective abstraction. Piaget regarded the acquisition of mathematical knowledge to be associated with the latter. Reflective abstraction refers to the construction of logico-mathematical structures by a learner during the process of cognitive development (Dubinsky, 1991a). The two features of this concept are: (a) It has no absolute beginning but appears at the very earliest ages in the coordination of sensori-motor structures, and (b) It continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction (Dubinsky, 1991b).

We define the following four concepts that are used in APOS theory of conceptual understanding (Bowie, 2000):

- Action: an action is a repeatable physical or mental manipulation that transforms objects
- Process: a process is an action that takes place entirely in the mind.
- Object: the distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made.
- Schema: a schema is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema could aid students to "... understand, deal with, organise, or make sense out of a perceived problem situation" (Dubinsky 1991a, p.102).

According to Sfard (1991) abstract mathematical notions (concepts) can be conceived in two fundamentally different ways: as *processes* (operationally) or *objects* (structurally). In APOS theory action and process can be regarded as operational conceptions, while object and schema are structural. Reification "... is an act of turning computational operationals into permanent object-like entities" (Sfard 1995, p.16). The development of mathematics often proceeds by taking processes as operators and then turning them into objects. Examples of processes as operators are counting, calculating using a formula (for example, using the nth term of a sequence to generate successive terms) and differentiating; while examples of resulting objects are numbers, algebraic expressions (for example, the nth term of a sequence) and the first derivative of a function. Therefore reification, which refers to a transition from an operational to a structural mode of thinking, is a basic phenomenon in the formation of a mathematical concept since it brings the concept "... into existence and thereby deepens our understanding" (Sfard 1994, p.54). Both operational (procedural) and structural thinking are important in mathematics - both contribute to the hierarchical structure of algebra, which is used to represent mathematical concepts symbolically.

This paper adopts the following five kinds of construction in reflective abstraction explained by Dubinsky (1991b):

- Interiorisation: the ability to apply symbols, language, pictures and mental images to construct internal processes as a way of making sense out of perceived phenomena. Actions on objects are interiorized into a system of operations
- Coordination: two or more processes are coordinated to form a new process
- Encapsulation: the ability to conceive a previous process as an object
- Generalisation: the ability to apply existing schema to a wider range of contexts
- Reversal: the ability to reverse thought processes of previous interiorized processes

The study focused on advanced mathematical thinking required for the concepts of monotonicity and boundedness of infinite real sequences. This falls under the domain of APOS theory.

Methodology and subjects

The method adopted four stages: (a) Design of worksheet, (b) Facilitation of group-work, (c) Capture of written responses, and (d) interviews. The first two stages were influenced by social constructivism, since learning without participation is in contradiction with the recommendations of constructivists such as Von Glasersfeld (1984), Cobb (1994), Confrey (1990) and Steffe (1992). Von Glasersfeld (1984) argued that reflective ability is a major source of knowledge in all levels of mathematics. This implies it is important for students to talk about their thoughts to each other (Brijlall & Maharaj, 2006) and their lecturer. The worksheets were designed to facilitate these interactions, so that students could support each other to construct new mathematical knowledge. The students involved were undergraduate teacher trainees from the University of KwaZulu-Natal. They pursued a module on Real Analysis in their final year. This module, which included elementary topology of the real line, involves the learning of concepts in set theory, relations and functions, cardinality, countability, denseness, convergence and other related ideas.

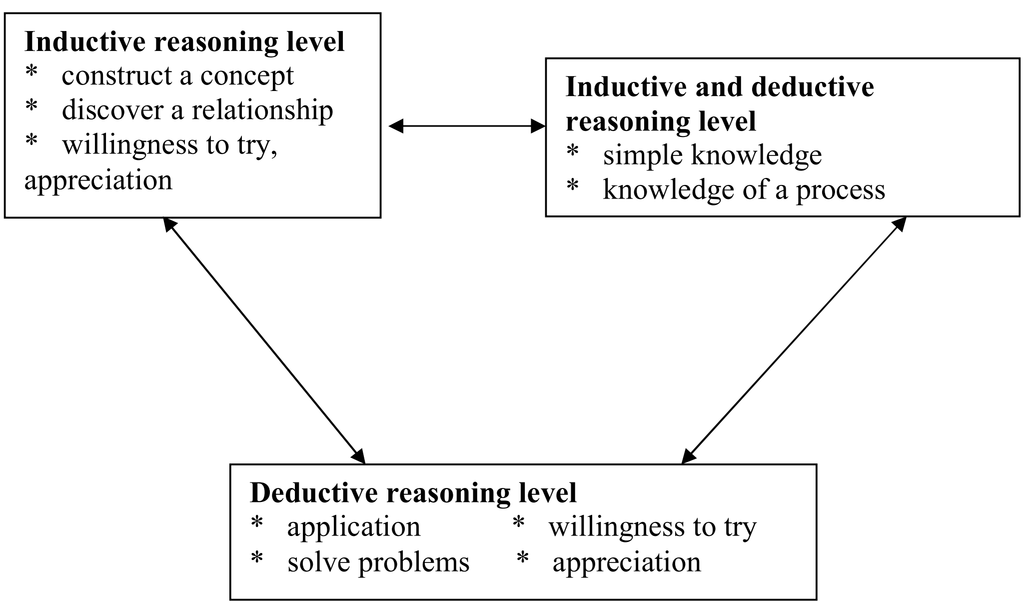
Instrumentation

Design of worksheet

Worksheets were designed in accordance with ideas postulated for a guided problem solving linear model suggested by the work of Cangelosi (1996). His work modeled how meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and affective domains. The model has the following three levels/phases: (a) Inductive reasoning (conceptual level), (b) Inductive and deductive reasoning (simple knowledge and knowledge of a process level), and (c) Deductive reasoning (application level). However, our experience over time has indicated that this model should be adapted (see Figure 1) since there is continually an inter-play between inductive and deductive reasoning (Maharaj, 2007). They are continuously present and constantly following each other in mathematical thinking. For example, in an inductive process very often a preliminary 'generalising' step is reached. A conclusion or the finalisation of an inductive part is the beginning of the deductive part. Therefore generalising at each of the different levels implies that the deductive mode of reasoning comes into play. In creating constraints for the examples and non-examples in the guided worksheets implemented in this research we kept in mind the characteristics of boundaries as suggested by Mason & Watson (2004). Boundary examples are those examples that "distinguish having and not having a specified property" (Mason & Watson, 2004, p.9)

For the design of the worksheets inductive learning activities were used to construct the concepts of monotonicity and boundedness of real sequences. These activities had the following stages (a) sorting [examples and non-examples] and categorising, (b) reflecting and explaining the rationale for categorising, (c) generalising by describing the concept in terms of attributes [that is, what sets examples of the concept apart from non-examples], and (d) verifying and refining [the description or definition is tested and refined if necessary].

Figure 1: A Guided Problem Solving Teaching Model



To determine the monotonicity of a sequence required knowledge of a process. Here an algorithm had to be formulated which required the calculation of successive terms of a sequence, and then comparing them. Stages in the two structured worksheets were designed to facilitate conceptual understanding by focusing on the following which form the framework for APOS theory: action, process, object and schema.

Data Collection Procedures

Facilitation of group-work

A fourth year class of 23 students were presented the worksheets and engaged with the activities individually for approximately fifteen minutes. [The reason for this was that we felt they first needed to work individually, so that later in the group context they could make constructive contributions to their respective groups.] Thereafter, when constructing the concepts they worked in seven groups (five comprising of three members and two with four). Each group, after discussing and reaching a collective decision, documented their thoughts for presentation to the class. After all the presentations a class discussion including the response of the lecturer led to the accepted definitions of the two concepts.

Capture of written responses

Each student was given a guided activity sheet. When they were in groups they were provided a separate worksheet which required the collective group response to the activities. The following five instructions appeared on the worksheets: (a) Complete each worksheet on an individual basis. (b) You are now required to form groups of three or four. (c) Now discuss your findings within the group to reach consensus. (d) Write down a collective response and elect a leader to discuss with class. (e) Finally conclude findings as a class with your tutor. These worksheets were then collected by the researchers for analysis of student thinking.

Verification interviews

Whilst analyzing the written feedback clarity was required for certain responses. Questions were designed specifically to further probe and clarify such responses. The relevant questions were posed to the respective group leaders during one on one interviews.

Pre-knowledge

At this stage in the course students covered fundamental concepts on set theory, methods of proof and logic, relations, and basic ideas on topology of the real line. Hereafter, they were engaged in issues relating to real infinite sequences. They covered the definition of a real sequence and its convergence as:

- A real infinite sequence $(x_n)_{n=1}^{\infty}$ is a function $f: \mathbb{N} \to \mathbb{R}$ defined as $f(n) = x_n$
- A real infinite sequence $(x_n)_{n=1}^{\infty}$ converges to L if for every $\in > 0$, there

exists a natural number *N*, such that $n > N$ implies $|x_n - L| < \in$.

Post-knowledge

During the guided problem solving activity they developed the following definitions:

- A real infinite sequence $(x_n)_{n=1}^{\infty}$ is called monotonically increasing (decreasing) if $x_{n+1} > x_n$ $(x_{n+1} < x_n)$ for all $n \in \mathbb{N}$.
- A real infinite sequence $(x_n)_{n=1}^{\infty}$ is bounded above (below) if there exists a real number *M* (*m*) such that $x_n \leq M$ $(x_n \geq m)$ for all $n \in \mathbb{N}$.

A real infinite sequence $(x_n)_{n=1}^{\infty}$ is monotone or monotonic if it is either increasing *or* decreasing and bounded if it is bounded above *and* below. The aim hereafter is to prove the theorem "Every monotonic bounded real infinite sequence is convergent". It is much easier to show that convergence implies boundedness. That this converse is not necessarily true can be demonstrated by the bounded sequence $((-1)^n)_{n=1}^{\infty}$ which is not convergent. The added condition of monotonicity allows for the truth of the converse, since a monotonic bounded infinite real sequence is convergent. This theorem then will be treated as a *final object*. The construction of the two concepts discussed in this paper required application of actions and finite processes. At each culminating step of these processes objects were obtained. We end at this theorem as a last step and thus envisage it as a final object (Dubinsky et al, 2005).

Presenting of Findings and Discussion

Monotonic Sequences

The following extract from the worksheet, modeled on the construction of a concept at the inductive reasoning level (Cangelosi, 1996) to promote the phases in APOS theory, indicates the task based on examples and non-examples which the students engaged with:

A] Sorting

The following infinite real sequences are called ***monotonically increasing***.

1. $(2^n)_{n=1}^{\infty}$ 2. $(2n+1)_{n=1}^{\infty}$ 3. $\left(\frac{n+1}{2}\right)_{n=1}^{\infty}$ 4. $(2n-1)_{n=2}^{\infty}$

The following infinite real sequences are not ***monotonically increasing.***

1. $(2^{-n})_{n=1}^{\infty}$ 2. $\left(\frac{2}{2n+1}\right)_{n=1}^{\infty}$ 3. $\left(\frac{1}{2n}\right)_{n=1}^{\infty}$ 4. $((-1)^n)_{n=1}^{\infty}$

B] Reflecting and explaining

After interrogating the above examples and non-examples of monotonic increasing infinite real sequences, explain why one would categorize them as such.

……………………………………………………………………………………

C] Generalizing the description of ***monotonically increasing*** infinite real sequences

Now write out a statement which you would adopt to describe (define) an arbitrary infinite real sequence $(x_n)_{n=1}^{\infty}$:

……………………………………………………………………………………

Table 1 summarizes the seven group responses in reflecting and explaining, and generalizing the concept monotonically increasing sequences. Characterization of coded categories is as follows: (a) *none* - was used for no response, (b) *inadequate* - codes an incorrect or unclear response, (c) *partial* - codes gaps in description, and (d) *complete* - codes a mathematically correct response.

Table 1: Results on constructions for monotonically increasing (n = 7).

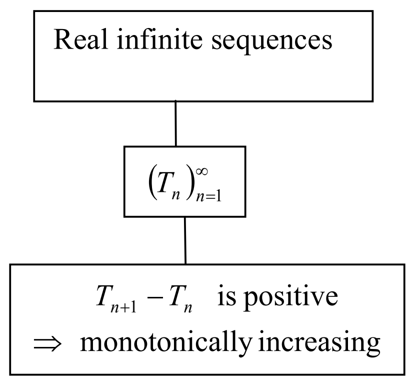
Stages	Number of group responses			
	None	Inadequate	Partial	Complete
Reflecting and Explaining	1	0	5	1
Generalizing	1	2	3	1

It was observed that five of the seven groups when reflecting and explaining constructed a partial understanding of this concept (see Table 1). Examples of such constructions stated by two of the groups were as follows:

> Group E: *The sequence always increase which imply that term (T_{n+1}) minus T_n will give a positive value and in monotonic increasing infinite real sequences there will be no variable in the denominator*.
>
> Group F: *A monotically increasing sequence is a sequence that approaches ∞ from the negative side as $n \to \infty$. In other words, as $n \to \infty$, x_n increases in value to ∞.*

The correct use of the symbols T_{n+1} and T_n to explain an increasing sequence implies that group E viewed the general term of a sequence as an object. Also note that this group devised an algorithm as a test for monotonically increasing sequences. Here reflective abstraction involved coordination, since the two or more processes applied to the individual terms of sequences were coordinated to form a new process to detect (a) firstly if they were monotonically increasing sequences, and (b) then generalized this. Further they had a schema for detecting monotonically increasing sequences (see Figure 2), since they were able to manipulate the objects (terms of the sequence) by internal processes. However, the exclusion of variables in the denominator indicates an incomplete understanding. This misconception arose because the four illustrative examples excluded such cases. The response of group F was coded as such due to the phrase *from the negative side*, since approaching positive infinity is from the side of positive numbers and the terms of the sequences (in question) were all positive.

Figure 2: Possible monotonically increasing schema for group E



Bounded Sequences

The following extract from the worksheet, modeled on the construction of a concept at the inductive reasoning level (Cangelosi, 1996), indicates the task based on examples and non-examples which the students engaged with:

A] <u>Sorting</u>

The following are examples of infinite real sequences which are ***bounded below:***

1. $\left(2^{n}\right)_{n=1}^{\infty}$ 2. $\left(2n+1\right)_{n=1}^{\infty}$ 3. $\left(\frac{n+1}{2}\right)_{n=1}^{\infty}$ 4. $\left(2n-1\right)_{n=1}^{\infty}$

The following are examples of infinite real sequences which are <u>not</u> ***bounded below:***

1. $\left(-\frac{n+1}{2}\right)_{n=1}^{\infty}$ 2. $\left(1-2n\right)_{n=1}^{\infty}$

The following are examples of infinite real sequences which are ***bounded above.***

1. $\left(2^{-n}\right)_{n=1}^{\infty}$ 2. $\left(\frac{2}{2n+1}\right)_{n=1}^{\infty}$ 3. $\left(\frac{1}{2n}\right)_{n=1}^{\infty}$ 4. $\left((-1)^{n}\right)_{n=1}^{\infty}$

The following are examples of infinite real sequences which are not ***bounded above:***

1. $\left(2^{n}\right)_{n=1}^{\infty}$ 2. $\left(2n-1\right)_{n=1}^{\infty}$

B] Reflecting and explaining

After interrogating the above examples and non-examples of bounded (above or below) infinite real sequences, explain why one would categorize them as such.

…………………………………………………………………………………

C] Generalizing the description of ***bounded above/below*** infinite real sequences

1. Now write out a statement which you would adopt to describe (define) an arbitrary infinite real sequence $(x_n)_{n=1}^{\infty}$ as bounded above:

………………………………………

2. Now write out a statement which you would adopt to describe (define) an arbitrary infinite real sequence $(x_n)_{n=1}^{\infty}$ as bounded below:

…………………………………………

D] Verifying and refining

Check whether the following are ***bounded above/below*** infinite real sequences by applying the above definition.

1. $(\log_2 n)_{n=1}^{\infty}$ 2. $\left(2^{n}\right)_{n=1}^{\infty}$ 3. $\left(\log_{\frac{1}{2}} n\right)_{n=1}^{\infty}$

Table 2 summarizes the seven group responses in reflecting and explaining, and generalizing the concept, bounded sequences. Characterization of coded categories is as for Table 1.

Table 2: Results on group constructions for bounded below/above (n = 7).

Stages		Number of group responses			
		None	Inadequate	Partial	Complete
Reflecting and Explaining	Bounded above	0	5	2	0
	Bounded below	0	5	2	0
Generalizing	Bounded above	0	4	1	2
	Bounded below	0	4	1	2

A striking observation is that the majority of the responses were categorized as inadequate. Such responses for bounded sequences under reflecting and explaining led to inadequate generalizations. This suggests that during the process of constructions of concepts, inadequate reflecting and explaining is likely to produce inadequate generalizations. Group B was the exception to this assertion. Their response for reflecting and explaining is given below:

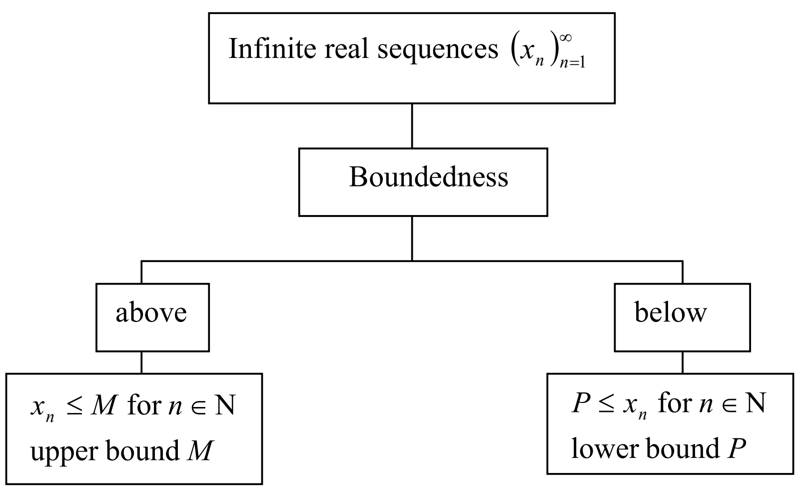
A sequence X_n is bounded if $\{X_n / n \in \mathrm{N}\}$ is a bounded set. For a sequence to be bounded below it has X_1 as it's minimum value where it is a lower limit. For a sequence to be bounded above it has X_1 as it's maximum value where it is an upper limit.

The response regarded the first term of a sequence as a determining factor in concluding the type of boundedness of a sequence. In generalizing the description of bounded above this group's response was:

If $x_n \leq M$ for $n \in \mathrm{N}$ M is a constant (independent of n) we say that the sequence $(x_n)_{n=1}^{\infty}$ bounded above and M is called an upper bound.

This group correctly generalized the dual of this statement for the concept bounded below. (Note that these responses were in accordance with the definitions of bounded above/below as discussed in the section on post knowledge.) Such responses suggest that group B were able to formulate schema which summarized tests for boundedness (see Figure 3).

Figure 3: Possible boundedness schema for group B



Further, during reflective abstraction, generalization as explained by Dubinsky (1991b) was involved in the construction. This is so since it seems that students were able to apply existing schema to a wider range of contexts by including upper and lower bounds. The use of terminology such as upper/lower bound and upper/lower limit was spontaneously introduced by the students in group B. Despite inadequately framing their responses (for the reflecting and explaining processes) in the construction of the concept of boundedness, their generalization displayed that they conceptualized sequences as objects. This supports the claim of an inter-play between inductive and deductive reasoning, within the framework of the guided problem solving teaching model (see Figure 1).

Group E gave a partial response in reflecting and explaining the concept of boundedness, yet were able to correctly generalize. This is more likely to occur than the exception which was illustrated above during the analysis and discussion of the response by group B.

It was observed that six out of the seven groups when identifying the boundedness of a sequence believed that it is bounded either above or below, and not both. For example, Group E when discussing the second sequence in the table wrote:

$$n. > 1$$
$$\therefore 2^n > 2^1$$
$$\therefore xn. > x.1$$
$$\therefore \left(2^n\right)_{n=1}^{\infty} \text{ is bounded below.}$$

We note the misuse of notation in line 3, despite the correct conclusion of the sequence being bounded below. No investigation into the sequence being bounded above was done.

Table 3 summarizes the seven group responses for the verifying and refining stages during further abstraction in construction for the concept of boundedness.

Table 3: Results on verifying and refining boundedness (n = 7).

Sequences	Number of group responses			
		None	Incorrect	Correct
$\left(\log_2 n\right)_{n=1}^{\infty}$	Bounded above	5	1	1
	Bounded below	2	1	4
$\left(2^n\right)_{n=1}^{\infty}$	Bounded above	5	1	1
	Bounded below	2	1	4
$\left(\log_{\frac{1}{2}} n\right)_{n=1}^{\infty}$	Bounded above	3	0	4
	Bounded below	4	2	1

Group B was the only group that investigated both the types of boundedness for each sequence. Their response for the second sequence in Table 3 was:

it is bounded below by 2 and not above.

This was a concise and apt response. Note that group B displayed understanding when generalizing the concept of boundedness and also conceptualized sequences as an object (see analysis and discussion for Table 2). In Dubinsky's stages of reflective abstraction the constructions of interiorisation, coordination, encapsulation and generalization were demonstrated by group B. This seems to intimate that these constructions are pre-requisites for successful applications.

We finally summarized the scores of these students in the final examination to gage their performance. The examination question paper had allocated 150 marks in total. The questions were based on section A (set theory and methods of proof – 47 marks), section B (relations and functions – 48 marks), section C (sequences - 35 marks) and section D (series – 20 marks). Twenty two candidates' scores were considered as one student was refused a duly performance certificate (which refused entry to the examination). The median for section C was 20 which meant a percentage of 57 as opposed to an overall median of 76 which implied a percentage of 50. We observed therefore a better performance by the majority of students in the section on sequences. Perhaps this could be attributed to the approach which required students to engage in collaborative learning via the use of worksheets, which for each concept facilitated action, process, object and schema (that is, APOS).

Conclusion

The structured worksheets; based on the examples and non-examples approach for constructing the concepts of monotonocity and boundedness of sequences; within the framework of APOS theory, had a positive impact. It seems that this was so because they encouraged group-work, which fostered an environment conducive to reflective abstraction.

The findings (as demonstrated by the analysis and discussion of the responses of group B) showed that students demonstrated the ability to apply symbols, language, and mental images to construct internal processes as a way of making sense of the concepts of monotonocity and boundedness of sequences. On perceiving sequences as objects students could apply actions on these objects which were interiorized into a system of operations. The verifying and refining stages in the construction of the two concepts required a conceptualization of these concepts as objects. This conceptualization enabled the formulation of new schema which we envisage to be applied to a wider range of contexts. The concept of boundedness of sequences should now lead to the construction of the definitions of *supremum* (least upper bound) and *infimum* (greatest lower bound) of infinite real sequences.

References

Adler, J. (2002). Inset and mathematics teachers conceptual knowledge in practice. In P*roceedings of the 10 th Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology.* (Vol. II, pp.1-8).

Barkley, E.F., Cross, K.P. & Major, C.H. (2005). Collaborative Learning Techniques. John Wiley & Sons Inc. USA.

Bowie, L. (2000). *A Learning theory approach to students' errors in a calculus course,* Pythagoras no. 52. Centrahil:Pretoria.

Brijlall, D. (2005). A Real Analysis course for BEd undergraduate students. *Presented at the 1st Africa Regional Congress of the International Commission of Mathematical Instruction*. University of Witwatersrand.

Brijlall D., Maharaj A. & Jojo Z. M. M. (2006). The development of geometrical concepts through design activities during a Technology education class. *African Journal of Research in SMT Education*. Volume 10(1), 37-45.

Cangelosi, J. S. (1996). *Teaching Mathematics in Secondary and Middle School: An Interactive Approach, Second edition*. New Jersey: Prentice Hill.

Cobb, P. (1994). Where is the mind? Constructivist and socio-cultural perspective in mathematical development. *Educational Researcher*, *23*(7), 13-20.

Confrey, J. (1990). A review of research on student conceptions in mathematics, science and programming. *Review of Research in Education*, *16*, 3-55.

Department of Education., (1999). Norms and Standards for Educators, Pretoria, National Department of Education.

Department of Education. (2003). Revised national curriculum statements grades 10 – 12 (schools) mathematics, Pretoria, National Department of Education.

Dubinsky, E. (1991a). Reflective Abstraction in Advanced Mathematical Thinking. In: Tall, D. (ed), *Advanced Mathematical Thinking*. The Netherlands: Kluwer.

Dubinsky, E. (1991b). Constructive Aspects of Reflective Abstraction in Advanced Mathematical Thinking. In: Steff, L. (ed), *Epistemological Foundations of Mathematical Experience*. Yew York: Springer Verlag

Dubinsky, E. & Harel, G. (1992). *The nature of the process conception of function.* In: Harel, G. & Dubinsky, E., The concept of functions: aspects of epistemology and pedagogy, MAA Notes series, USA: Mathematical Association of America.

Dubinsky, E., Weller, K., McDonald, M.A. & Brown, N. (2005). Some historical issues and paradoxes regarding the concept of infinity: an APOS analysis: Part *2. Educational Studies in Mathematics*. 60, 253 – 266.

Maharaj, A. (2005). Investigating the Senior Certificate Mathematics Examination in South Africa: Implications for Teaching. Phd thesis In Mathematics Education: Unisa.

Maharaj A. (2007). Using a task analysis approach within a guided problem-solving model to design mathematical learning activities. *Pythagoras*. Number 66(Dec), 34-42.

Mason,J, .& Watson, A. (2004). *Getting students to create boundary examples.* http:/www.bham.ac.uk/ctimath/talum/newsletter/talum11.htm.

Mwakapenda, W. (2004), Understanding student understanding in mathematics, *Pythagoras*, No 60, 28-35.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as two sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (1994). Reification as the Birth of Metaphor. *For the Learning of Mathematics*, 14(1), 44-55.

Sfard, A. (1995). The Development of Algebra: Confronting Historical and Psychological Perspectives. *Journal of Mathematical Behaviour*, 14, 15-39.

Steffe, L.P. (1992). Prospects for alternative epistemologies in education. In L.P. Steffe (Ed.), *Constructivism in education* (pp.86-102). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Vidakovic, D. (1996). Learning the concept of Inverse Function. *The Journal of Computers in Mathematics and Science Teachin,* ***15*****(3), 295-318.**

Vidakovic, D. (1997). Learning the concept of inverse function in a group versus individual environment. In Dubinsky, E., Mathews, D. & Reynolds, B., (Eds.), *Readings in Cooperative Learning* , MAA Notes No 44, 173-195.

Von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp.75- 92). New York: Norton.

Wu,H. (2003). *Preservice professional development of mathematics teachers.* http://www.math.berkeley.edu/~wu/.

AUTHORS

Aneshkumar Maharaj is a mathematics lecturer at the University of KwaZulu-Natal. His research interests focus on how to promote the teaching and learning of mathematics, and advanced mathematical thinking. He has taught senior secondary mathematics, trained mathematics teachers and lectured to university mathematics students.

Deonarain Brijlall is a lecturer in mathematics education at the University of KwaZulu-Natal. His research focuses on learning and teaching theories in mathematics education, with particular emphasis on students' conceptual understanding.