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Continuities and discontinuities for mental models -
A source for difficulties with the multiplication of fractions

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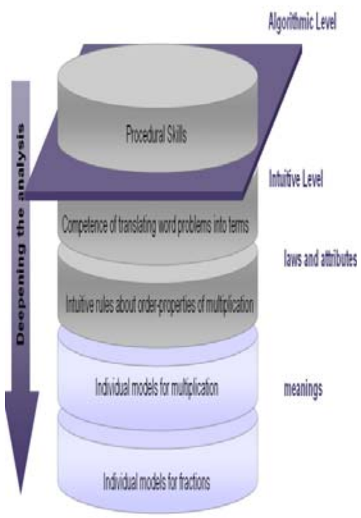
Difficulties with fractions and decimals as an issue of research

Various empirical results: enormous difficulties with fractions (and decimals)
(e.g. Hasemann 1981, Streefland 1984, Barash/Klein 1996, Aksu 1997,....)

- less with algorithmic skills (like multiplication of fractions)
- more with solving word problems and with understanding



Different approaches for

locating and explaining the difficulties

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Approach 1: Emphasis on (mental) models (Grundvorstellungen)

(Fischbein/Nello/Marino 1985, Usiskin 1991, Greer 1994, vom Hofe/Blum et al. 2005)

repeated addition
(3x5 means 5+5+5,
i.e. 3 wands of 5m length in a row)

part-of-interpretation
(2/3 x 5/2 means 2/3 of 5/2)

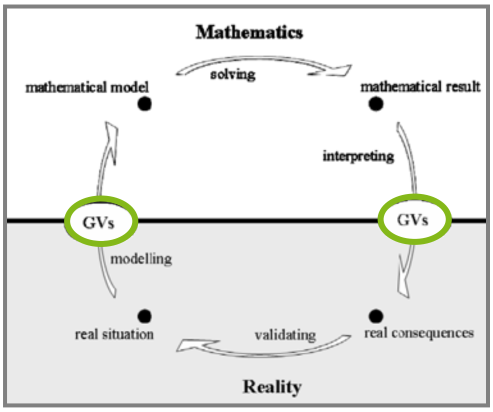
area of a rectangle

acting across different quantities
(kg x price per kg)

multiplicative comparison
(twice as much)

scaling up and down
(3x5 means 5cm is
stretched three times as much)

combinatorial interpretation
(3x5 as number of combining
3 shirts + 5 trousers)



(vom Hofe et al. 2005)

mental model as a "meaningful interpretation of a phenomenon or concept" (Fischbein 1989)

models as „use meaning" (Usiskin 1991)

Approach 2: Discontinuities between natural and fractional numbers

many authors (Streefland 1984, Hartnett/Gelman 1998, Brousseau 1980, Winter 1999, ...)

most influential: conceptual change approach
(Lehtinen 2004: ICME-10 plenary, Posner et al. 1982, Lehtinen/Merenluoto/Kasanen 1997, Stafylidou/Vosniadou 2004)

Aspect	Natural numbers	Fractional numbers
Cardination	a number is the answer to the question "How many?"	a fraction can describe parts of a whole, quotients, ratios, proportions, …
Symbolic representation	one number	two numbers and a line
	unique relation between number and symbolic representation	existence of many fractions representing the same fractional number
Ordering	supported by the natural numbers' sequence (counting on)	not supported by the natural numbers' sequence
	existence of a successor (discreteness)	there is no unique successor or a unique preceding number (density)
	no number between two different numbers	density: infinite many numbers between each two numbers
Operations		
Addition–Subtraction	supported by the natural numbers' sequence	not supported by the natural numbers' sequence
Multiplication	multiplication makes the number bigger	multiplication makes the number either bigger or smaller
Division	division makes the number smaller	division makes the number either smaller or bigger

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Synthesis: (Dis-)Continuities in the mental models (for multiplication)

(Prediger 2008 JLI, similar Greer 1994)

Natural numbers	Fractions
repeated addition (3x5 means 5+5+5, i.e. 3 wands of 5m length in a row)	???
area of a rectangle (3x5 is the area of a 3cmx5cm rectangle)	area of a rectangle (2/3 x5/4 is the area of a 2/3 cm x 5/4 cm rectangle)
????	part-of-interpretation (2/3 x 5/2 means 2/3 of 5/2)
acting across different quantities (kg x price per kg)	acting across different quantities (kg x price per kg)
scaling up (3x5 means 5cm is stretched three times as much)	scaling up and down (2/3 x 5/2 means 5/2 cm compressed on 2/3 of it)
combinatorial interpretation (3x5 as number of combining 3 shirts + 5 trousers)	????

Research Questions of our Study

- What mental models for the multiplication of fractions do students activate in Grade 7 and 9 (age 12 and 14)?
- What kind of situations can they describe by a multiplicative term?
- Is there empirical evidence that “discontinuous models” are more difficult than those which can be continuously transferred from natural numbers to fractions?

Research Design

Test design and sample:

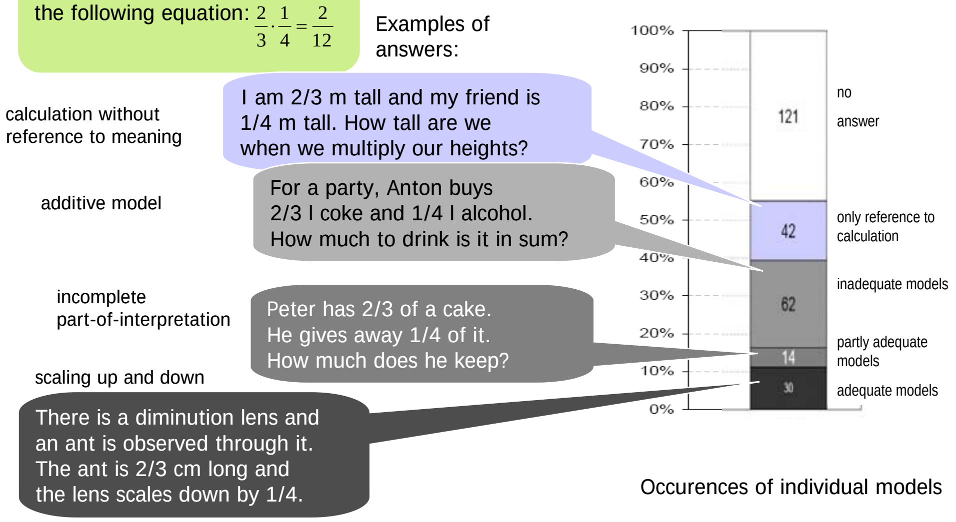
- paper and pencil test (60 minutes)
- 269 students in Grade 7 and 9 (12-14 years) in “Gymnasien” (schools for higher achieving 40%)

Data analysis

- quantitative evaluation of students’ answers in a points rationing scheme
- qualitative coding of answers on open items (interrater agreement: Cohen’s kappa = 0.81 - 0.94)
- calculating and comparing occurences of codes

Which mental models do students activate?

Item 6: Find a word problem that can be solved by means of the following equation: $\frac{2}{3}\cdot\frac{1}{4}=\frac{2}{12}$



Occurences of individual models

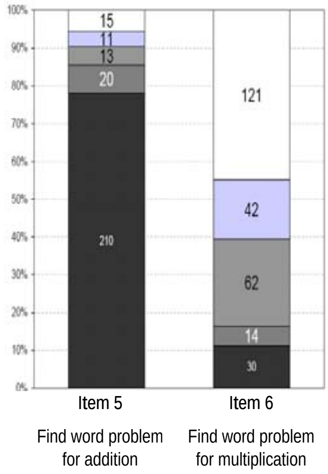
Which mental models do students activate?

much more difficulties for interpreting multiplication than for addition

(background: less discontinuities for mental models of addition!)

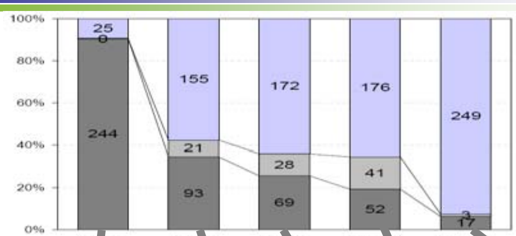
Item 5: Find a word problem that can be solved by means of the following equation: $\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$

Item 6: Find a word problem that can be solved by means of the following equation:



Comparison of occurences of individual models

What kind of situations can they describe by multiplication terms?



Item 8a repeated addition

Every child eats on average 2/10 g of mashed potatoes.
How can we calculate what 15 children would eat?
☐ 2/10 – 15 ☐ 2/10 : 15
☐ 2/10 · 15 ☐ 15 · 2/10
☐ none of these, but

Item 7a across quantities

One kilogram tangerine costs € 1.50. Kate wants to buy ¾ kg. How can she calculate her price to pay?
☐ 1,5 – ¾ ☐ 1,5 : ¾ ☐
☐ 1,5 · ¾ ☐ none of these but

Item 9a part-of-interpretation

How can we calculate ¾ of 36? ☐ ...

Item 11 scaling down

An African elephant has a body height of 3.60 m. Anna has a model of the elephant which is scaled down to 3/40 of the original body height.
Give the height of the model elephant.
Solve the task with one single operation.

Item 10 part-of fraction (picture)

Colourize ¾ of the rectangle (given in picture).

Colourize now 2/5 of these 3/4 with another colour.

Give the fraction that describes the part of the rectangle that is double coloured now.

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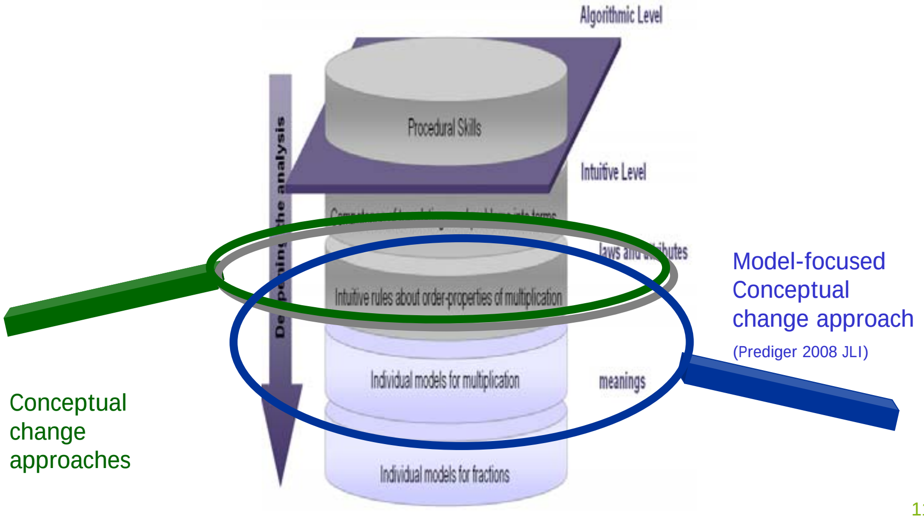
Conclusion

- Only models and discontinuities together give explanations for difficulties!
- Find the right level of discontinuities for focus on conceptual change



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The full paper is available under http://www.mathematik.uni-dortmund.de/~prediger/veroeff/08-ICME11-TSG10-Fractions.pdf