

# Knowledge for teaching mathematics through inquiry

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## Introduction and abstract

Teacher knowledge is often suggested to be a strong indicator of students' opportunities to learn deep mathematical content (Hill, Rowan, & Ball, 2005). Debate remains, however, how to develop teachers' knowledge for teaching mathematics and avenues to assess the knowledge that teachers draw upon for classroom practice. A study conducted in Australia was designed to gain insight into primary teachers' learning and evolving practices in implementing multiple curriculum units using mathematical inquiry. This paper draws on Eraut's (1994) framework for knowledge use to report on mathematical practices displayed by the teachers in solving ill-defined problems. By examining characteristics of the teachers' approaches in initial experiences with inquiry and those taken by teachers with significant experience with inquiry, insight into the critical role that generalisability of practice within ongoing experiences plays for developing and assessing knowledge for teaching.

## Literature review

Mathematical knowledge for teaching has long been reported as limited to incomplete procedural knowledge (Ma, 1999). Rather than emphasise shortfalls, research must look at ways to assess and support teachers' knowledge development. Previous studies have often focused on tangible factors such as experience, qualification, and formal content knowledge that have not satisfactorily captured the kinds of knowledge that influence student learning (Hill et al, 2005). Science education advocates development of teachers' knowledge "requires learning essential science content through the perspectives and methods of inquiry" (p. 55, National Research Council, 1996). Although the National Council of Teachers of Mathematics (2000) does not specifically name inquiry as a mathematical practice, it promotes teacher learning of mathematical content in contexts consistent with inquiry: worthwhile mathematical tasks, mathematical discourse, and *doing* mathematics. Ball (1996) concurs that teachers need opportunities to build connections with mathematics as learners, further arguing (Ball, 2002) that understanding mathematics goes beyond traditional mathematics coursework to include the development of *mathematical practices*. "Being able to justify claims, using symbolic notation efficiently, and making generalizations are all examples of mathematical practices. Such practices are important to both learning and doing mathematics. Their absence can hamper mathematical learning" (p. 24). If teachers are to develop this knowledge through experiences as learners, research needs to focus on elaborating and theorising these experiences.

Eraut (1994) distinguished between two issues critical to analysis of professional knowledge: theoretical knowledge and generalisability of practical knowledge. However, he cautioned that, theoretical ideas usually cannot be applied ‘off-the-shelf’: their implications have to be worked out and thought through. The busy professional ... is unlikely to find time for that. Thus the functional relevance of a piece of theoretical knowledge depends less on its presumed validity than on the ability and willingness of people to use it. This is mainly determined by individual professionals and their work context, but is also affected by the way in which the knowledge is introduced (p. 43).

Eraut argues that development of practical knowledge for use emerges through generalisation from cases and experiences. He contends that by studying this generalisation process as it occurs, and making the process more explicit for professionals, their knowledge development can be realised.

Eraut’s framework for knowledge use (developed from Broudy et al, 1964) includes four levels:

- *Replication*: knowledge acquisition and rehearsal occurs within a context similar to its use;
- *Application*: knowledge use requires unrehearsed transfer to an unfamiliar context;
- *Interpretation*: knowledge is purposefully transformed and redeveloped as it is applied, relying on in-the-moment judgement, justification, and consideration of its feasibility;
- *Association*: knowledge use is developed through extensive experience; its application is highly intuitive and often relies on imagery that cannot necessarily be articulated explicitly.

Most problems in school mathematics fall into the first level; rarely does it address mathematical problems requiring interpretation or association. Eraut advocates using ill-defined problems<sup>1</sup> (Reitman, 1965) in professional learning to provide opportunities to experience ways that traditional approaches fall short prompting a need to re-evaluate knowledge strategies in new ways. Eraut notes that unlike well-defined problems, ill-defined problems have no solution or multiple solutions, and small changes in the problem definition often require large changes in the solution.

## **Context and Study Design**

The research question addressed in this paper is: *What characteristics of mathematical knowledge use are exhibited by primary teachers with varying levels of expertise in teaching inquiry-based mathematics?* The research was developed as a Design Experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), used to gain insight into complex classroom environments. In a design experiment, the researcher works to understand and simultaneously improve the context under investigation through multiple iterations of data collection and analysis. Four teachers from suburban River State School (RSS) were involved in the initial phase (2006-2007) of the study, with eighteen teachers in phase two (2007-2009) from River and rural Bush State Schools (BSS). (See Makar, in press-a; in press-b for other results published elsewhere.) Teachers took part in three days of professional development each year in which they shared experiences and concerns, experienced solving ill-defined inquiry problems, and designed inquiry-based curriculum units. These professional learning seminars were recorded and portions transcribed for later analysis.

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<sup>1</sup> Ill-defined problems are those for which the problem definition is ambiguous or has many open constraints.

During the year, teachers were expected to teach four inquiry-based mathematics units. Lessons were videotaped and teachers interviewed to study evolving experiences over multiple iterations.

This paper focuses on mathematical knowledge that teachers displayed in solving ill-defined problems during learning seminars. Four episodes are reported [cut back to two episodes in this shortened version of the paper] as teachers collaboratively work through tasks (described below).

Table 1 summarises the tasks, teachers involved, and levels of experience with inquiry.

*Table 1: Episodes reported*

Experience with inquiry	Chair task	Newspaper task
Initial	<u>*Episode 1: March 2006</u> 4 RSS teachers, grades 4-5	<u>Episode 3: September 2007<sup>†</sup></u> 2 RSS teachers, grades 5-6 4 BSS teachers, grades 4-7
	<u>Episode 2: September 2007<sup>†</sup></u> 5 BSS teachers, grades 1-3 1 BSS teacher, grade 7	
Experienced		<u>*Episode 4: August 2007</u> 4 RSS teachers <sup>θ</sup> , grades 3-5

\*Two teachers in Episode 4 were also in Episode 1 in the previous year. <sup>†</sup>Episodes 2 & 3 are in the full version of the paper. <sup>θ</sup>For one teacher in Episode 4, this was her initial experience (attending with the experienced group because of a scheduling conflict); her vocal contributions do not appear in the episode.

## Tasks

Mathematical knowledge utilised to solve each problem was not pre-determined and multiple possible solution strategies were within reach of primary school teachers. However, the problems required application of knowledge in ways that likely differed from their experiences.

*Chair task.* This ill-defined problem (adapted from Economopoulos, Mokros, & Russell, 2004), asked teachers to design an ergonomic chair for their classroom. The problem was intentionally vague in order to trigger discussion to mathematise and define task parameters. It was hypothesised that the teachers would need to consider relationships in relevant body measurements that would relate to the comfort of the chair (different types of chairs were made available).

*Newspaper task.* This ill-defined problem (adapted from Rubin & Mokros, 2004) asked teachers, ‘What fraction of the newspaper is ads?’ It was hypothesised that teachers would need to debate what counts as an ad as well as to plan and implement an approach to solve the problem. Potentially, teachers might collect data on the coverage (area) of ads in newspapers (provided).

## Initial experiences with open-ended problems

Three episodes [two in this abridged paper] are described here in which teachers engage with ill-defined problems in their first experience with inquiry. In these problems, note teachers’ strategies, the mathematics that is used and how, and the explicitness of their meta-analysis of the problem.

## Episode 1 (Chair task, 96 minutes)

Initial discussions (first 5 minutes) by the teachers centred on consideration of elements they believed that chair manufacturers likely included when designing an ergonomic chair:

- Average height of the population
- Distance from the knee to the floor
- Femur length
- Height of the small of the back
- Consideration of age and posture
- Tilt of chair and ability to recline
- Height of desk
- Angle of chair (unspecified which part of the chair)

Over the first hour, the teachers discussed characteristics and varying comfort levels of different chairs in the room and collected multiple measurements from each other (body heights, femur lengths, knee-to-floor lengths, chair seats—heights, widths, and depths, and height of chair backs), occasionally debating the need for particular measurements.

Natasha: Alright, do we need this measurement, the depth of the chair?

Kaye: I think we need it, because if you've got a short chair, you're going to be uncomfortable. ... This is actually cutting into my legs here (indicating the seat of a chair she is sitting on), so to sit here for too long [would be uncomfortable] ... I do believe we need this measurement (indicated femur). [10 minutes into the activity]

During this time, the researcher observed and occasionally probed teachers' thinking. After 20 minutes, the teachers began to feel overwhelmed:

Kaye: There's so many measurements that affect [this]! ... Does age affect them? Does someone who is tired or who has [back] degeneration, might prefer a different chair, or-

Josh: -Sex?

Carla: Too many factors...

Natasha: ... but which factors are the most important?

The discussion digressed to chairs at school and other chairs they have encountered in their lives until they realised that they were getting off track:

Kaye: Our focus question has probably (flown out the window). ...

Carla: ... Ok, let's think about what we've found so far, ... where we're up to. [25:00]

The teachers summarised their findings so far and preferences for the different chairs, trying to associate these preferences to the measurements of their bodies. They continued to debate aspects of chairs (armrests, back angles), discussed classroom environments, and took measurements, but often without a clear purpose. They expressed concern about their inability to find a constant ratio to work with and turned again to discussions of padding, back rests, and types of chair legs to consider. The researcher intervened to help them consider the information they had so far. Natalie suggested they collect data.

Natasha: I think what we really need to do is to get a whole school, or several schools, and look at the ratios of children. ... We'll need at least a couple hundred kids at each grade, I'd say a minimum of a hundred at each grade level. [45:00]

The teachers considered collecting body measurement data and survey opinions about different chair preferences from some of the university students outside the session room. The researcher

[KM in the transcript] intervened again to push the teachers to reach a decision about what information they'd actually need to collect. After another five minutes, they digressed again.

- KM: Are you feeling sometimes like your head is spinning? Like there's too much?  
Carla: Just where to start?  
Josh: Too many variables!  
Natasha: Too many parameters!  
KM: So the more variables, the more complex the situation becomes. So try to think about simplifying it, so maybe you're down to just a handful of things that you will either ask or measure. [50 minutes]

Teachers continued discussions, again remeasuring themselves and chairs and settled on three variables to collect from passing university students—knee-to-floor height, preferred chair (of three provided), and reasons for preference. They collected data on 23 university students that walked by the room and continued to debate chair preferences. The time ran out before reaching a conclusion.

*Summary.* After initially focusing on the context of the problem and factors that influence comfort in chairs, the teachers moved quickly into measuring a myriad of elements that might be needed. They found the task overwhelming at times, unsure of what measurements and characteristics to include and spent a lot of time measuring, discussing, and remeasuring. During the problem, they attended to the need to measure carefully and drew on mathematical knowledge about measurement, ratios, averages, and sampling. Although they found the task challenging, most of the conversation remains on task and focused on the mathematics that might be needed. One element that appeared to be missing in the problem was a vision of a problem structure, an aspect of ill-defined problems identified as one of the most challenging (Reitman, 1965) and central to Eraut's upper levels of knowledge use.

*[Episodes 2 & 3 removed from this abridged version of the paper]*

## **Teachers experienced with mathematical inquiry of ill-defined problems**

Finally, we turn to examining the approach taken by a group of teachers who had been teaching mathematical inquiry for two years and inspect their approach to the newspaper task. Note that two of these teachers, Natasha and Kaye, also took part in the chair task (Episode 1) the previous year.

### **Episode 4 (Newspaper task, 70 minutes)**

Within the first minute of the activity, the teachers raised the need to clarify what they thought should be the criteria for an ad and suggested ideas for measuring and calculating the fraction of the paper that was ads. Without prompting, they began debating possible approaches.

- Kaye: So the question is, do you have to go through the entire paper? Or can you take a sample?  
Elise: Or would you do a whole paper and see if you can use that inferentially? [3:00]

They left that question and spent several minutes debating how to decide what counts as an ad.

- Kaye: Can we just do, perhaps, generalise? Can we just do what is not an article written by a journalist? Everything else becomes advertisements for the purpose of the task we are doing ...

that's our justification. Because there are so many other bits ... we could spend the whole session doing that.

Their recognition of the power of generalising their definition of an ad using criteria provided some insight into their knowledge of content as inclusive of justifiable mathematical practice and suggests that the teachers could foresee the importance of defining ambiguities in ill-defined problems. The teachers carried on and did a meta-analysis of the process of justification and whether debating what counts as an ad is a mathematical process or a literacy process. After five minutes, the researcher brought their attention back to the task.

KM: So, what fraction of the paper is ads?

Kaye: I'd like to do what ratio is ads.

Elise: Are you going to go for space or headings?

Natasha: The easiest way to do it would be simply to count: article, article, ad. But I think it's-

Elise: The space thing is more valid, isn't it? ... Are we going for space or number?

Kaye: She [KM] just said 'what fraction', so-

Natasha: I'm inclined to go with space. ... But obviously we're not going to count the border because every page has a border. Or are we just going to disregard the border? [24:00]

The group reviewed their definition of an ad using criterion discussed in the previous exchange and clarified a few ambiguous cases (e.g., letters to the editor) and then discussed measurement strategies for coming up with the areas of ads in relation to the page and agreed on levels of precision. The group recognised the enormity of the task and considered ways to simplify it.

Elise: So do we have to do every page?

Natasha: We're going to have to, [so] then you'll have to [just] estimate.

Kaye: Is it worth, maybe just doing a selection of pages and seeing if they are hugely different?

Elise: Then could we do the first three pages and last three pages [and compare them]?

Kaye: Yes, something like that. [29:00]

They took measurements of each and compared them; dissatisfied with the outcome, the group searched for other approaches, always discussing the feasibility and validity of each and how they would assess and justify it. Throughout their discussions, the teachers' self-monitoring to check the validity and justification of approaches was an element that set them apart from the initial groups.

The four teachers split into two pairs, each taking on a different approach and using a different sample of pages. After fifteen minutes, they compared their results and were satisfied this time with the similarities in outcome. Elise indicated satisfaction to generalise their findings to the whole paper, saying, "I would feel happy that that's enough for inferences" [57:00].

The group uses the criteria of inferences—a topic that had come up during the year as they were working through curriculum units in their classrooms that included inquiry with data—to assess their level of confidence in their findings. 'Enough for inferences' to them seemed to indicate sufficient confidence in the outcome for that paper, but not necessarily to extend more generally to other copies of the paper. The teachers ended the activity with a meta-analysis of their sampling technique and how that affected their level of confidence in the outcomes, admitting they weren't quite satisfied with the weight of evidence they had by sampling only eight pages of the paper.

*Summary.* Several aspects of their knowledge use were apparent in examining the approach to the task displayed by the teachers with experience teaching inquiry-based mathematics. For one, their ability to envision the problem allowed them to recognise quite quickly the need to be explicit about clarifying ambiguities in the task. Second, they felt the need to justify and validate their approaches without prompting from the researcher, ensuring that the decisions they made would hold up under scrutiny. Finally, they frequently engaged in a meta-analysis of the problem and solution processes to reflect on ways that their experiences might generalise for classroom practice. These elements suggest that the teachers had acquired a sense of vision about ill-defined problems in mathematics that provided them with generalised pathways to approach their solution.

## **Discussion**

Eraut (1994) argues that “learning knowledge and using knowledge are not separate processes but the same process. The process of using knowledge transforms that knowledge so it is no longer the same knowledge” (p. 25). To clarify their relationship, he maps out two kinds of professional knowledge central to our analysis: theoretical knowledge, based on written knowledge available in books; and practical knowledge which emerges out of generalisations of cases and experiences. Central to knowledge development is its use, which he argues takes on four levels: replication, application, interpretation, and association.

The teachers in Episodes 1-3 who were just beginning their journey demonstrated several commonalities in their knowledge of mathematics in an inquiry-context. For one, they often approached the problem task without first mathematising the problem or determining the relationship between the task and the mathematics that was useful for the task. In the chair problem (Episodes 1 and 2), for example, a great deal of time was spent on discussing contextual factors that did not work towards a particular solution to the problem. The teachers struggled to see how the mathematics would help them to design an ergonomic chair, suggesting difficulty in transforming their theoretical knowledge into a more flexible and usable form. In all three of the episodes, the teachers relied on fairly simplistic mathematics and often did not feel compelled to dig deeper beyond a surface approach. Their ability to connect the mathematical knowledge they knew with the context under study was often tenuous, frequently relying on basic mathematical knowledge, but not on the mathematical practices (Ball, 2002) to justify, quantifying relationships, or generalise their findings beyond the local problem. Their ability to recognise that they were ‘finished’ with the task typically required intervention by the researcher to press for deeper analysis.

In Episode 4, the teachers who had gained a great deal of experience in the teaching and learning of mathematics through inquiry took a distinctively different approach to their problem task. They appeared to possess a vision of ill-defined problems that provided them with an understanding of

the need to clarify and mathematise the problem. Their solution did not simply entail an outcome, but articulated its justification and assessment of its validity. Separating the theoretical knowledge from mathematical practice was not always clear, as exhibited in the teachers' engagement with meta-analysis of processes. Rather than find identified complexities overwhelming and work to dismiss them, they appeared to envision them as part of the character of ill-defined problems and worked to simplify the approach to take the problem complexity into account.

In drawing again on Eraut's (1994) framework for knowledge use, the teachers in their initial experiences struggled to move within or beyond a level of application of their knowledge. The unfamiliarity and ambiguity of the problem context sparked frustration and uncertainty that proved difficult to cope with. Alternatively, the teachers who had developed significant experiences in teaching mathematical inquiry through the use of ill-defined problems demonstrated not only resilience in successfully managing the problem context, but also drew on more interpretive and associative levels of knowledge use through their expectation to include justification, assessment of validity, vision of problem processes, and engagement with meta-analysis. In this way, the framework may provide opportunities to assess teachers' mathematical knowledge use in ways not possible through traditional measures. Further research may provide greater insight into the potential of Eraut's framework for analysing teacher knowledge beyond this exploratory study.

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